

ESD-TDR-65-75

**ESD RECORDED COPY**

RETURN TO  
SCIENTIFIC & TECHNICAL INFORMATION DIVISION  
(ESTI), BUILDING 1211

**ESTI PROCESSED**

☐ DDC TAB ☐ PROJ OFFICER  
☐ ACCESSION MASTER FILE  
☐ \_\_\_\_\_

DATE \_\_\_\_\_

ESTI CONTROL NR

AL

46314

CY NR

OF

CYS

**Technical Note****1965-16****A Program for the Estimation  
of Power Spectra***ESRL*

C. R. Arnold  
E. Korngold

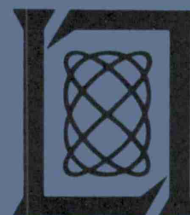
**21 May 1965**

Prepared under Electronic Systems Division Contract AF 19(628)-500 by

**Lincoln Laboratory**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts

*AD0616677*

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-500.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
LINCOLN LABORATORY

A PROGRAM FOR THE ESTIMATION OF POWER SPECTRA

*C. R. ARNOLD*

*Group 28*

*E. KORNGOLD*

*PRESS Field Site  
Kwajalein*

TECHNICAL NOTE 1965-16

21 MAY 1965



## ABSTRACT

A straightforward and general program of spectral analysis has been written in FORTRAN II for the IBM 7090 and is described in this report. The analysis is valid for sequences of equidistant data sampled from realizations of second order stationary stochastic processes. Alternatively, the program may be used to estimate the transfer function gain characteristic of a linear system on the basis of its sampled output.

The Inputs to the program consist of the data sequence to be analyzed and FOUR control parameters.

The Output consists of listings of the estimated values and of three CALCOMP plots of

- (i) The sample auto covariance (ACV) functions.
- (ii) The power spectral density (PSD).
- (iii) A log-log plot of the PSD.

The estimated PSD is consistent, being a periodogram smoothed with Hanning weights.

After a brief introductory discussion in Section I, Section II proceeds with a sketch of the analytic background, and a discussion of the parameters critical to a power spectral analysis. Section III is a description of the program and has some sample estimates. Operating instructions are given in Section IV as well as a complete description of the outputs. Section V describes an alternate use of the program. Section VI points to possible modifications to tailor the program more nearly to the individual requirements of a prospective user.

References are listed in the back of the report and are noted in the text by indicating the reference number in square brackets. For someone anxious to use the program it may be best to read the section on operating instructions (Section IV) first along with Section II C which described the significance of the program parameters.

Accepted for the Air Force  
Stanley J. Wisniewski  
Lt Colonel, USAF  
Chief, Lincoln Laboratory Office



## TABLE OF CONTENTS

ABSTRACT	iii
I. INTRODUCTION	1
II. ANALYTIC BACKGROUND	2
A. Continuous Case	4
B. Discrete Case	5
C. Choice of Parameters	7
1. DT and the Sampling Theorem	8
2. K and PSD Resolution	8
3. Sample Size and S/N Ratio	9
4. Summary	9
III. PROGRAM DESCRIPTION	9
A. Inputs	11
B. Computations	11
C. Sample Outputs	14
IV. OPERATING INSTRUCTIONS	18
A. Usage	18
B. Deck	18
C. Card Formats	21
D. Outputs	21
E. Limitations	22
V. APPLICATION: ESTIMATION OF TRANSFER FUNCTION GAIN CHARACTERISTICS	22
A. Linear Systems	23
B. Nonlinear Systems	25
VI. GUIDELINES TO POSSIBLE MODIFICATIONS	30
A. Input	30
B. Calculation of Sample Autocovariance	30
C. Periodogram Smoothing	31
D. Output	31

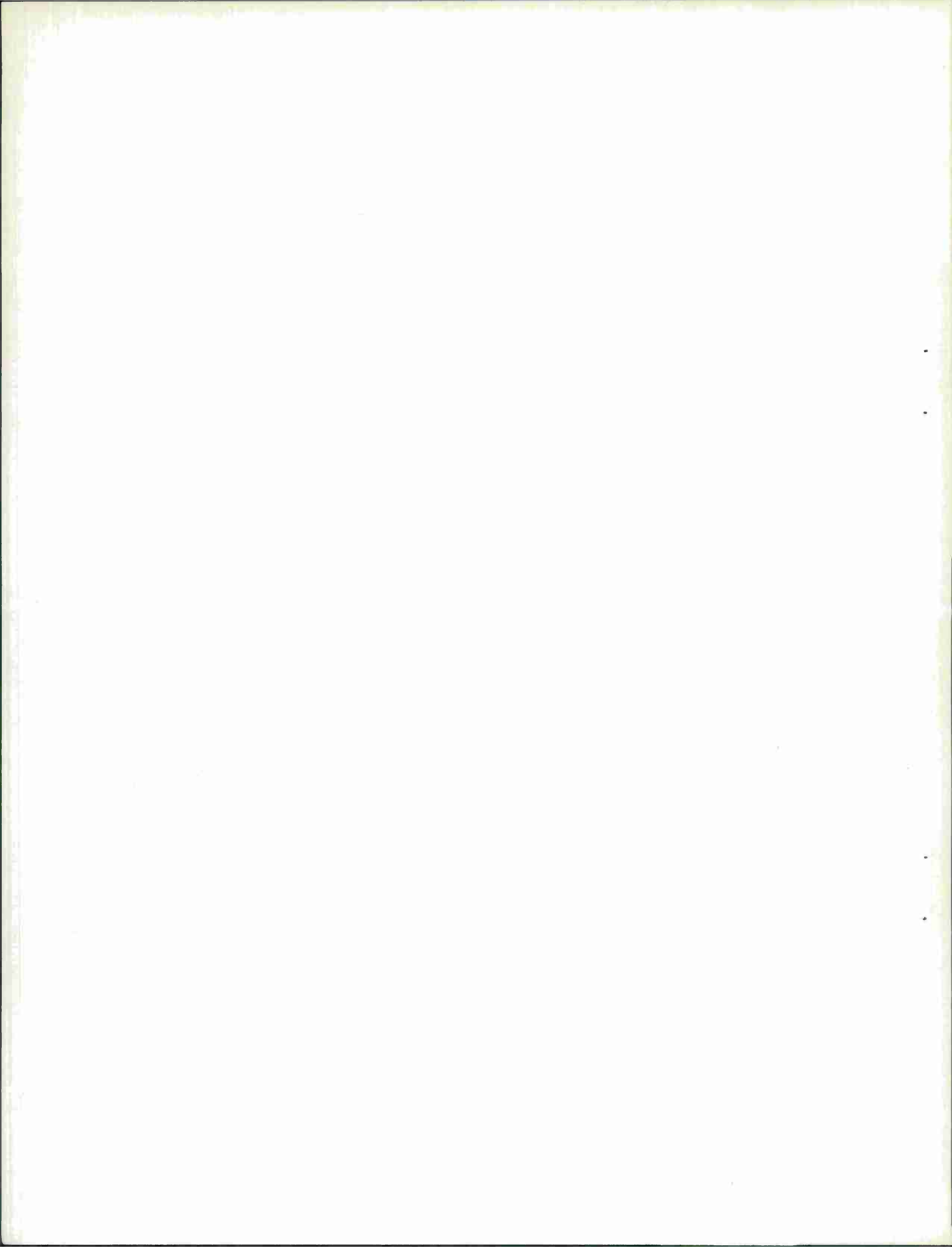




TABLE OF CONTENTS (Cont'd)

APPENDIX A	FINITE FOURIER COSINE TRANSFORMS	32
APPENDIX B	PROGRAM LISTING	34
APPENDIX C	SAMPLED LISTED OUTPUTS	40
REFERENCES		46



## I. INTRODUCTION

The last 15 years have seen a renewed interest in the estimation of power spectra as a tool in the periodic analysis of data. One of the primary concerns has been to find suitable modifications of periodogram analysis to render consistent (in the statistical sense) the estimates of power spectra (or power spectral density), and the principal device used to obtain consistency has been smoothing of the periodogram.<sup>†</sup> A multitude of smoothing sequences (or equivalently lag windows) have made their appearance and are treated in varying detail in Refs. [2, 3, 4]. There still is a good deal of controversy over the problem of choosing the parameters of the smoothing process (bandwidth or equivalently the truncation point in the corellogram) to insure an optimal estimate, let alone over initial agreement on a criterion of optimality.

The basic dilemma is that given only a sample of data, and no background information on either the process generating the data or the recording device used in gathering the data, there exists no unambiguous method of obtaining an estimate of the Power Spectrum which may be regarded as best, see Ref. [5].

In the absence of clearcut guide lines, we have settled on a moderately simple and practical procedure (periodogram smoothing with Hanning Weights) and have incorporated it into a program in such a way that, should another smoothing sequence appear overwhelmingly preferable, the modification to the program can be made most trivially.

The key note of the program (the AMPSDE Program: Any Man's Power Spectral Density Estimator) is its simplicity and ease of use. The number of available options has been reduced to a minimum without curtailing the program's general usefulness as an exploratory device. As previously mentioned, and as described more fully below, minor modifications are easy to make by an interested user in the light of his own personal requirements, principally in the areas of input, output and choice of smoothing windows.

---

<sup>†</sup> An introductory exposition of the general problems encountered in the analysis of data from the point of view of its frequency content can be found in Ref. [1].

In a similar vein, the outputs of the program are provided without option, to insure a minimum of complexity in use and maximum ease in interpretation of results. For each set of data to be analyzed, three plots<sup>†</sup> are invariably provided:

- (i) A normalized Auto-Covariance (ACV) plot.
- (ii) A Power Spectral Density(PSD) plot on linear scale.
- (iii) A Power Spectral Density(PSD) plot on a log-log scale.

Along with the graphical outputs, the program prints a number of lists of the estimated values which are more fully detailed later in the report.

The inputs to the program consist of four constants, in addition to the data sequence to be analyzed. Several cases may be stacked in a single run.

The next section proceeds with the analytic background of the report, starting with a brief outline of the continuous case, its adaptation to discrete data and ending with a discussion of the parameters which were selected to control the program and their interpretation.

Section III is a detailed description of the program, with flow charts included wherever they clarify the presentation.

Operating instructions are given in complete detail in Section IV and a complete sample of annotated outputs is included.

An application of the program to the estimation of transfer functions is discussed in Section V.

Finally Section VI points out the ease with which the program may be modified.

Appendix A contains a brief note on discrete Fourier Cosine Transforms. A complete program listing is given in Appendix B. Appendix C exhibits a sample of listed outputs, while graphical outputs are illustrated in Figs. 3, 4 and 5.

## II. ANALYTIC BACKGROUND

The estimation procedure used in the program follows the simplest lines suggested in Ref. [3], but there, this simplicity is somehow imbedded in such a wealth of information that the clarity of the procedure is all but apparent.

---

<sup>†</sup> The graphical outputs are produced on a CALCOMP plotter.

The basic theorem states that if a process  $x_n$  has a summable<sup>†</sup> auto covariance sequence  $\gamma(k)$ ,  $k = 0, \pm 1, \pm 2, \dots$ , then its power spectral density exists and is given by

$$f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) \cos(k\omega).$$

All estimates  $\hat{f}(\omega)$  of  $f(\omega)$  which have been devised so far are ultimately of the form

$$\hat{f}(\omega) = \frac{1}{2\pi} \sum_{k=-N+1}^{N-1} h_T(k) \varphi_T(k) \cos(k\omega),$$

where  $\varphi_T(k)$  is an estimate of  $\gamma(k)$  based on the sample (dependence on sample size has been shown by the subscripted  $T = N\Delta t$ ) where  $n = 0, 1, 2, \dots, N$  and is generally taken as:

$$\varphi_T(k) = \begin{cases} \frac{1}{N - |k|} \sum_{n=0}^{N-|k|} (x_n - \bar{x})(x_{n+|k|} - \bar{x}), & k = 0, \pm 1, \pm 2, \dots, N-1 \\ 0, & |k| \geq N \end{cases},$$

and where  $h_T(k)$  is some covariance averaging kernel depending on the lag index  $k$  and possibly a sample size as well. Many different forms of  $\{h_T(k)\}$  have been suggested, but no clearcut method exists for choosing an optimal sequence in the absence of additional information on either the process or the sampling procedure.

---

<sup>†</sup> Summable  $\gamma(k)$  means  $\sum_{k=0}^{\infty} |\gamma(k)| < \infty$ .

Nor can an optimality criterion be chosen in vacuo, see Ref. [6]; for example, if we wanted to estimate the PSD at a single point  $\omega_0$ , we might seek to minimize

$$\mathcal{E} [\hat{f}(\omega_0) - f(\omega_0)]^2 .$$

If, on the other hand, one is interested in estimating the spectrum over an interval  $a < \omega < b$ , one might seek to minimize

$$\int_a^b \mathcal{E} |\hat{f} - f|^2 d\omega .$$

It is perhaps simplest to begin with a brief exposition of the continuous case and proceed to specialize to the situation which generally arises in practice:

#### A. Continuous Case

The simplest situation obtains when  $x(t)$  is a zero mean, stationary, ergodic, stochastic process whose auto covariance function (ACV)

$$(1) \quad \gamma(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau) dt = \gamma(-\tau)$$

admits a spectral representation in terms of  $f(\omega)$ , where

$$(2) \quad f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} \gamma(\tau) d\tau$$

$f(\omega)$  is called the power spectral density of the process and can be written as

$$(3) \quad f(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} x(t) e^{-i\omega t} dt \right|^2 .$$

Equations (1) and (2) express the Wiener-Khinchin theorem, while Eq.(3) shows why the periodogram appears to be a natural estimate of the PSD, (Ref. [1] ).

It may be worth placing the ACV function and the PSD of  $x(t)$  in evidence as a Fourier pair:

$$(4) \quad \gamma(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{i\omega\tau} d\omega$$

$$f(\omega) = \int_{-\infty}^{\infty} \gamma(\tau) e^{-i\omega\tau} d\tau \quad .$$

Since both  $\gamma(\tau)$  and  $f(\omega)$  are even functions, we may rewrite (4) as

$$(5) \quad \gamma(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) \cos(\omega\tau) d\omega = \frac{1}{\pi} \int_0^{\infty} f(\omega) \cos(\omega\tau) d\omega$$

$$f(\omega) = \int_{-\infty}^{\infty} \gamma(\tau) \cos(\omega\tau) d\tau = 2 \int_0^{\infty} \gamma(\tau) \cos(\omega\tau) d\tau \quad ,$$

where the second equality expresses  $\gamma(\tau)$  and  $f(\omega)$  as one-sided integrals.

#### B. Discrete Case

In order to apply digital methods we must sample  $x(t)$  at equidistant intervals  $\Delta t$  over a finite length of time  $T$ , to obtain a sequence

$$x_0, x_1, \dots, x_n, \dots, x_N$$

where

$$x_n = x(n\Delta t) \text{ and } N = T/\Delta t.$$

As a first step, in practice, it is necessary to remove the DC component (nonzero mean) from the data before computing the sample autocovariance,  $C_k$ . This is readily accomplished by replacing the original  $\{x_n\}$  by  $\{x_n - \bar{x}\}$ , where

$$\bar{x} = \frac{1}{1+N} \sum_{n=0}^N x_n .$$

One proceeds next to estimate the process auto covariance by the sample auto covariances.

$$(6) \quad C_k = \frac{1}{N - |k|} \sum_{n=0}^{N-|k|} x_n x_{n+|k|} \quad |k| = 0, 1, \dots, K$$

where  $K$  is the maximum lag index (necessarily  $< N$ ).

The obtained sequence of sample ACV's is normalized to

$$1 \equiv C_0 = \frac{C(0)}{C(0)} , \quad C_1 \equiv \frac{C(1)}{C(0)} , \dots, \quad C_K \equiv \frac{C(K)}{C(0)} ,$$

and a discrete finite cosine transform of the resulting sequence is taken, as shown in the following relations

$$(7) \quad \begin{aligned} P_0 &= \Delta t \left\{ C_0 + 2 \sum_{k=1}^{K-1} C_k + C_K \right\} \\ P_1 &= \Delta t \left\{ C_0 + 2 \sum_{k=1}^{K-1} C_k \cos \frac{k\pi}{K} + C_K \cos \pi \right\} \\ P_2 &= \Delta t \left\{ C_0 + 2 \sum_{k=1}^{K-1} C_k \cos \frac{k2\pi}{K} + C_K \cos 2\pi \right\} \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ P_r &= \Delta t \left\{ C_0 + 2 \sum_{k=1}^{K-1} C_k \cos \frac{kr\pi}{K} + C_K \cos r\pi \right\} \end{aligned}$$



$$\begin{array}{cccc}
 \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot
 \end{array}$$

$$P_K = \Delta t \left\{ C_0 + 2 \sum_{k=1}^{K-1} C_K \cos k\pi + C_K \cos K\pi \right\} .$$

Finally, the results may be smoothed by Hanning weights

$$U_0 = \frac{1}{2} (P_0 + P_1)$$

$$(8) \quad U_k = \frac{1}{4} P_{k-1} + \frac{1}{2} P_k + \frac{1}{4} P_{k+1} \quad k = 1, 2, \dots, K-1$$

$$U_K = \frac{1}{2} (P_{K-1} + P_K) .$$

#### C. Choice of Parameters

A cursory examination of the previous section (II, B) reveals that three parameters govern the process of generating PSD estimates. These are:

- (i) The number of data points in the sample (N+1).
- (ii) The sampling rate or the inter-sample time interval, (DT).
- (iii) The maximum lag index for calculation of auto covariances, (K).

For purposes of plotting and improved resolution in the vertical scale a fourth parameter has been included in the program:

- (iv) A scale factor by which all PSD estimates are multiplied.

We will discuss the significance of the scale factor in Section III where the calculations and output scaling performed by the program are described. We now turn to a discussion of the parameters which have analytical significance.

The most important single parameter of the program is the sampling rate as it controls the highest frequency component of the data which can be meaningfully identified. The maximum lag index determines the resolution in PSD obtainable by the program.

The number of data points determines the quality of the estimates, depending on the signal-to-noise ratio in the data and the data collection scheme.

1. DT and the Sampling Theorem - The sampling theorem states, in one of its forms, that if  $\Delta t$  seconds is the sampling interval in a sequence of data, then the frequency components of the process generating the data with frequency  $\omega > \frac{\pi}{\Delta t}$  radians /second cannot be distinguished from these with frequency in the range  $(0, \frac{\pi}{\Delta t})$  on the sole basis of the sampled values, see Ref.[ 7].

As a consequence it is best to choose  $\Delta t$  small enough, in practical situations, to insure that negligible power is contained by the process beyond the frequency  $\omega_{Ny} = \frac{\pi}{\Delta t}$  ( $\omega_{Ny}$  = Nyquist frequency). A high sampling rate, however, (small  $\Delta t$ ) means large quantities of data; to avoid unmanageable quantities of data it may be worthwhile passing the process through a low pass filter before sampling, if it is felt that there exist high frequency components in the process which are of no interest to the analyst.

In most situations, however, the analyst is presented with a collection of data where the sampling interval has already been determined. In this case, one must set the parameter DT equal to  $\Delta t$  and there is no way of determining whether there has been any aliasing of high frequencies, see Ref. [ 1].

The program automatically restricts itself to the basic interval from 0 to  $\frac{\pi}{\Delta t}$  in the calculation of estimated PSD.

2. K and PSD Resolution - Equation (7) of Section II.B can be rewritten in matrix form as

$$\underline{P} = \underline{M} \underline{C}$$

where  $\underline{P}$  is a (K+1) vector of estimate PSD values,  $\underline{M}$  is the coefficient matrix of cosines, and  $\underline{C}$  is the (K+1) vector of normalized sample ACV's.

These K+1 values of PSD are equally spaced over the interval  $[0, \omega_{Ny}]$ , so that if  $\Delta\omega$  is the angular frequency resolution desired, the analyst must set K such that

$$\frac{\omega_{Ny}}{K} \approx \Delta\omega \quad ; \quad \frac{f_{Ny}}{K} \approx \Delta f$$

or

$$(9) \quad K \approx \frac{\omega_{Ny}}{\Delta\omega} = \frac{\pi}{\Delta\omega(\Delta t)} \quad ; \quad K \approx \frac{f_{Ny}}{\Delta f} = \frac{1}{2\Delta f(\Delta t)} \quad .$$

3. Sample size and S/N Ratio - The accuracy of estimation is directly dependent on sample size. If the signal-to-noise ratio does not render the sampled values useless, a larger sample will in general yield better estimates. It is also advisable not to run the maximum lag at which ACV's are calculated too close to the sample size; indeed, Eq.(6) shows that as the lag index grows, the number of terms in the estimate of C, becomes smaller and hence the sample ACV's become poorer estimates of the process autocovariance for large lags. As a rule of thumb, a reasonable choice for maximum lag index might be 20% of the sample, so that, if K is determined on the basis of resolution requirements, one would let

$$N \approx 5K.$$

4. Summary - In recapitulation, after the sampling interval DT is set, the maximum frequency content of the sample is determined by

$$\omega_{Ny} = \frac{\pi}{\Delta t} \text{ radians/second or } f_{Ny} = \frac{1}{2\Delta t} \text{ cycles/second}$$

The analyst then, as a first approximation, may choose a resolution level by setting the maximum lag index K according to Eq.(9), and the sample size (N+1) to exceed 5 times the maximum lag. For high S/N ratios, K may go as high as 50% of sample length. In threshold situations, one may set K to be only 5% of the sample length in order to stabilize the autocovariance estimate.

### III. PROGRAM DESCRIPTION

A flow chart of the program is shown in Fig.1. A number in parentheses next to a box refers to the equation in the text relevant to the process described in the box. An encircled number corresponds to the output described in Section IV.D. Flow of control is marked in solid lines; flow of data is shown in dotted lines.

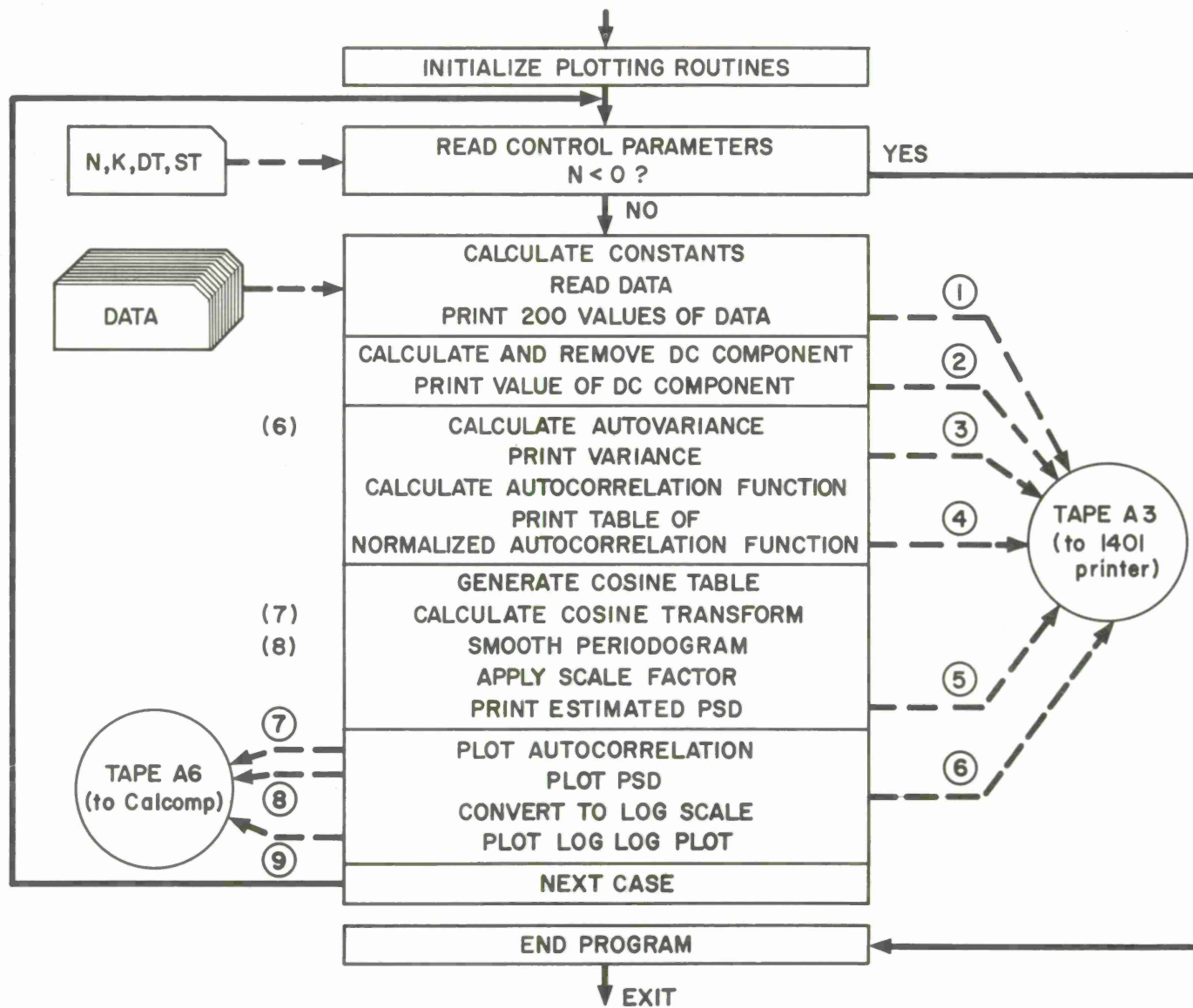


Fig. 1 Program Flow Chart

The program is written in FORTRAN II and makes use, for its graphical outputs, of a set of CALCOMP plotting subroutines which are documented in Ref.[8]. Apart from these subroutines, the program is completely straightforward as can be seen from the flow chart, or by consulting the listing given in Appendix B.

As has already been noted, the program is designed to spare the user from the necessity of choosing one out of an elaborate list of options. Instead, he will have to make judicious choices of the four parameters controlling the program.

We will now proceed to describe briefly several aspects of the program which warrant amplification.

#### A. Inputs

Four constants completely determined the course of the program. These are

- (i) The maximum sample index  $N$  equal to one less than sample size,
- (ii) The maximum lag index  $K$  for which sample autocovariances are calculated,
- (iii) The sampling interval  $DT$ ,
- (iv) A scaling factor  $SF$ .

The significance of these parameters (except for the 4th) has been examined in Section II.C.

#### B. Computations

The computations follow closely the procedure outlined in Section II.B. The dc. component is removed from the data before the frequency content of the data is examined. Autocovariances are calculated in the standard fashion. The cosine transform of the ACV function is taken and then smoothed, and final outputs are made while auxiliary outputs are generated at various points of the program.

A word is in order concerning the program's handling of Eq.(7), which can be rewritten as

$$\underline{P} = \underline{M} \underline{C}$$

where  $M$  is a  $(K+1) \times (K+1)$  matrix of cosines. A ingenious scheme avoids the necessity of calculating and storing  $(K+1)^2$  elements. Instead, only  $K+1$  cosine values need be calculated. Indeed, all the entries of matrix  $M$  are found among the  $K+1$  values

$$m_k = \cos \left( \frac{k\pi}{K} \right), \quad k = 0, 1, 2, \dots, K.$$

These are the only values computed by the program. The rows of matrix  $M$  are generated one by one by selecting the appropriate entries in the "cosine table," as described below. The cosine table is a sequence of  $2K+1$  elements formed by continuing the sequence  $m_k$  symmetrically, by reflection as shown in Fig. 2, for  $K=8$ , by way of illustration.

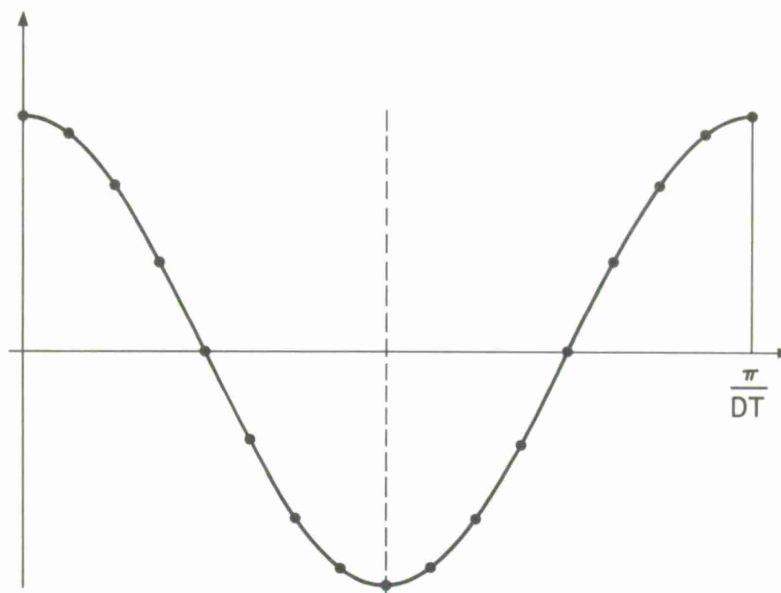


Fig. 2 Cosine Table

The rows of matrix  $M$  are formed by continuing periodically the sequence stored in the cosine table, or equivalently, by assuming that the cosine table is circularly stored so that the value following the last entry is the value stored in the first entry - and by selecting elements of  $M$  by the following scheme:

The first row of  $M$  contains  $K+1$  ones.

The second row of  $M$  contains the first  $K+1$  elements of the cosine table.



The third row of  $M$  contains the every second element of the cosine table, until a row or length  $K+1$  has been completed.

The fourth row of  $M$  contains every third element (Mod  $2K+1$ , the length of the table) of the cosine table, until a row of length  $K+1$  has been generated. And so on, until  $(K+1)$  cosine vectors have been generated and for each, the inner product has been formed with the autocovariance vector (sequence)  $C$ , the resulting  $(K+1)$  vector of inner products being the periodogram.

The PSD estimated by the program being a smoothed periodogram, a smoothing process occurs next, governed by Eq.(8).

Since the smoothed estimate has been calculated from the normalized autocovariances, the  $U_k$  of Eq.(8) are estimates of a normalized spectral density  $P_N(f)$ . This may be what is desired in that comparisons are sometimes best made upon a normalized spectrum, say

$$(10a) \quad \hat{P}_k = P_N(f_k) = \kappa U_k$$

where the factor  $\kappa$  is independent of  $k$  or of the sampled process, but may depend upon the possible overall frequency content ( i.e., on  $\Delta t$ ). Many times one will desire true levels given by

$$(10b) \quad \hat{P}_k = P(f_k) = \sigma^2 P_N(f_k) = \sigma^2 U_k$$

where  $\sigma^2$  is the sample variance. Finally, one may desire true levels for the process but must introduce some scaling constant due to the instrumentation and/or pickup sensors. For this case, one desires

$$(10c) \quad \hat{P}_k = \alpha \sigma^2 U_k$$

where as before  $\sigma^2$  is the sample variance and  $\alpha$  is some positive constant determined independently of the program.

The AMPSDE program is general enough to include all three of these options with the specification of a single parameter called the scaling factor (SF). Thus,

- (i) if  $SF = 0.0$ ,  $\hat{P}_k = \kappa U_k$  where  $\kappa = 1/\Delta t$
- (ii) if  $SF = 1.0$ ,  $\hat{P}_k = \sigma^2 U_k$  where  $\sigma^2 = \text{sample variance}$
- (iii) if  $SF = \alpha > 0$ ,  $P_k = \alpha \sigma^2 U_k$  where again  $\sigma^2 = \text{sample variance}$

The particular choice of  $\kappa$  in the first (normalized) case has been made so that the log-log plot of the spectra will always have sufficient detail for all ranges of frequency content. Also, if one desires the physical (one sided) power spectrum, one must use a scale factor of two (2.0) in the second case or  $2\alpha$  in the third case.

The estimated power levels usually have a wide range of variation; therefore the program provides for the calculation and plotting of the spectral estimate on a logarithmic (db.) scale. These db. levels being calculated from the equation

$$(11) \quad \hat{P}_{db}(f_k) = 10 \log_{10}(\hat{P}_k) \quad (\text{in db.})$$

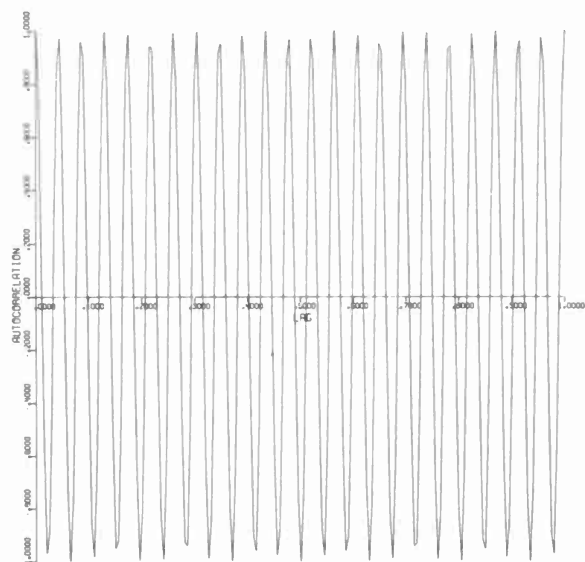
where the  $\hat{P}_k$  are the scaled smoothed estimates. The smoothed values which are negative or which yield power levels below -35 db. are set by the program to yield exactly -35db. The resulting db. power levels are also plotted on a logarithmic frequency scale, the plot having an overall range of 60 db. (from -35 to +25). The scale factor (SF) described above may also be used to obtain a better position of the spectral signature on the db. plot. The utility of this log-log plot in system studies is discussed in Section V.

The remainder of the program is taken up by plot generation and output processing.

### C. Sample Outputs

On the following three pages, we give sample output plots for three important classes of inputs. The actual input data had been previously synthesized from computer programs.





N = 2099  
K = 200

DT = 0.005  
SF = 2.0

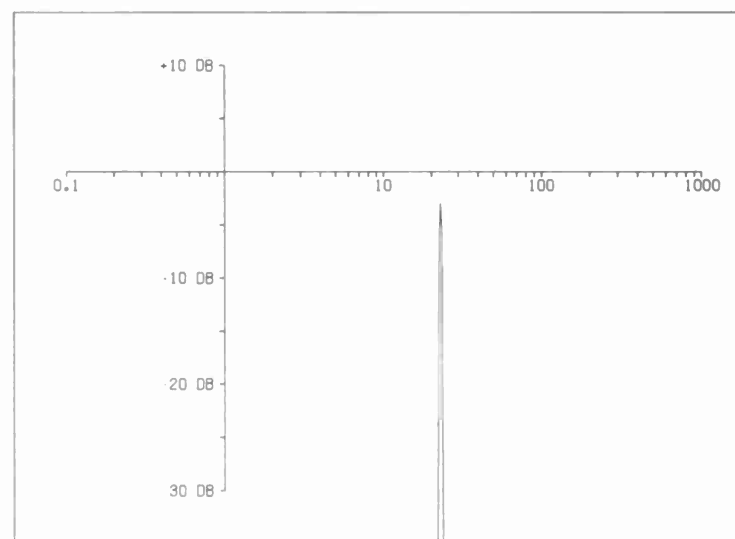
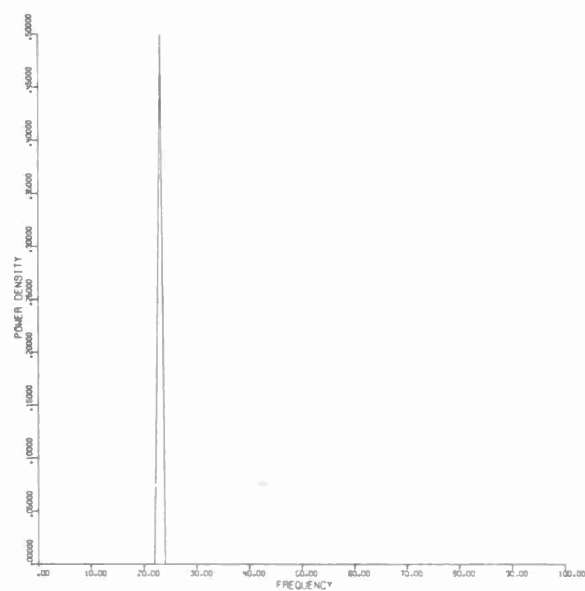
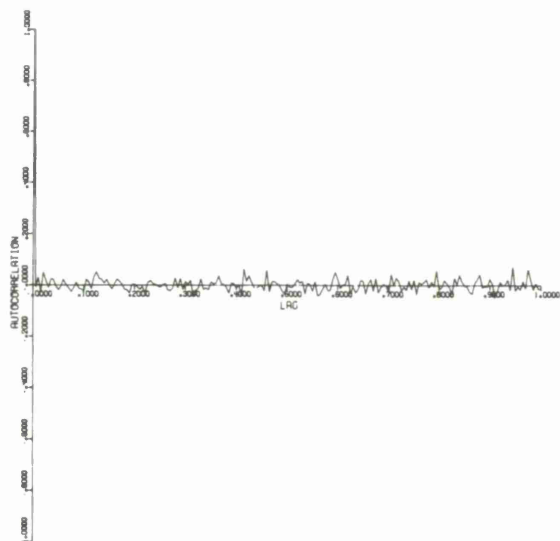


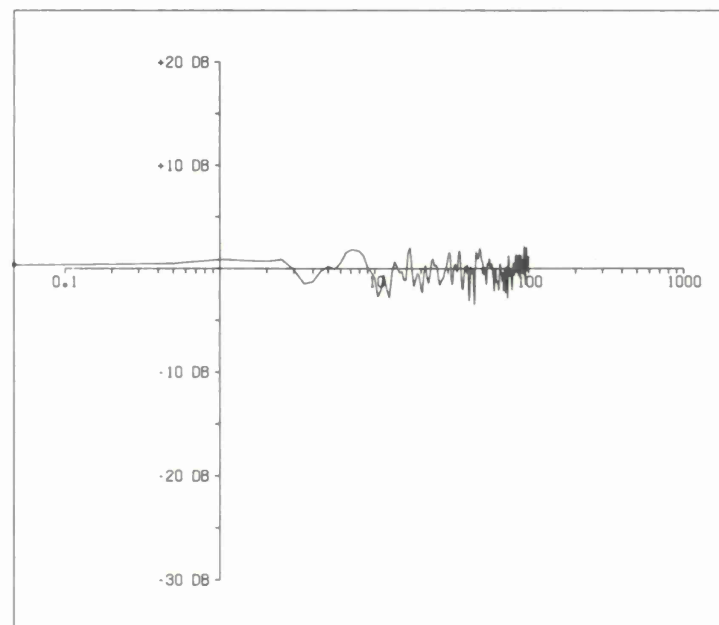
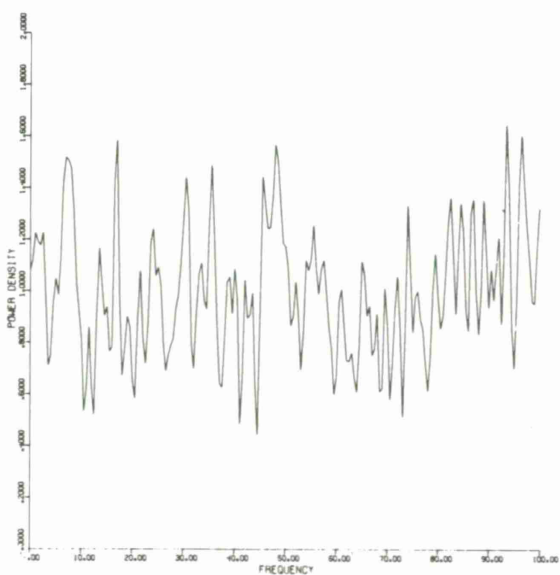
Fig. 3 Estimates from Sine Wave

c28-129



N = 2099  
K = 200

DT = 0.005  
SF = 200.0

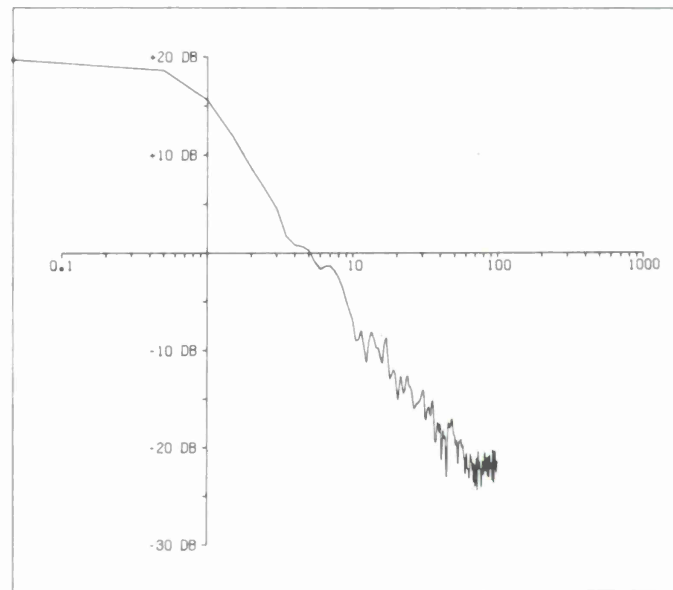
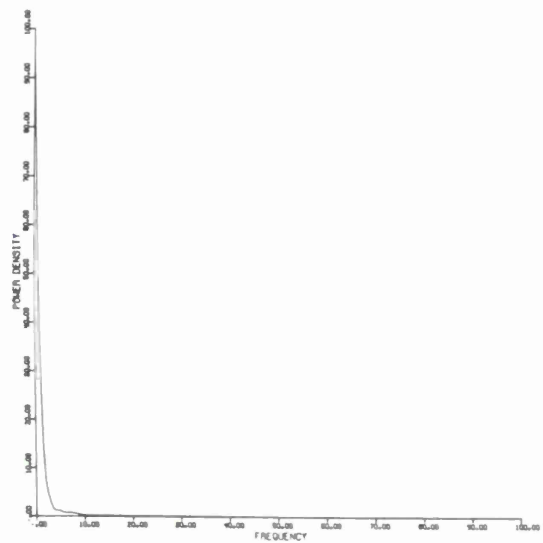
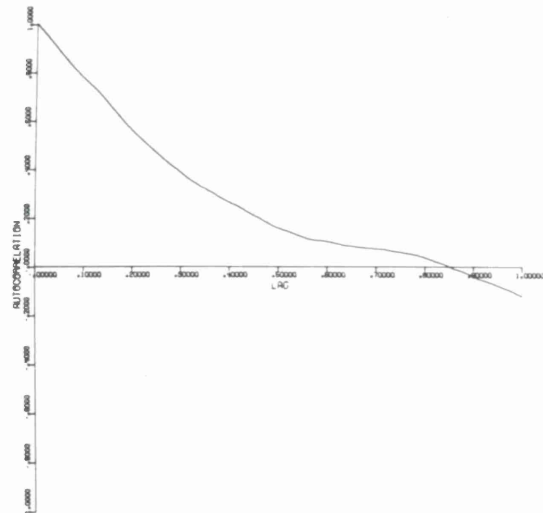


c28-132

Fig. 4 Estimates from White Noise

N = 2099  
K = 200

DT = 0.005  
SF = 0.0



c28-240

Fig. 5 Estimates of Colored Noise

For Fig. 3, the input was a 23 cycle, unit amplitude sine wave.

For Fig. 4, the input was a set of Gaussian distributed random numbers of approximately unit variance. These represent a sampled version of a white Gaussian noise process. With  $SF = 1/DT$ , the mean value of the spectral density is the zero db line.

For Fig. 5, the input sequence was the above noise sample but first shaped by a low pass filter. This sample is quite characteristic of real physical (colored) noise.

#### IV. OPERATING INSTRUCTIONS

The AMPSDE program is written in FORTRAN II for the IBM 7090 computer. To use the program, one needs:

- a) A program deck with CALCOMP subroutines
- b) One or more input data decks
- c) FORTRAN system tape
- d) Output tapes
- e<sup>†</sup>) CALCOMP plotter (off-line).

##### A. Usage

A usage diagram is shown in Fig. 6. As illustrated there, it is desirable to prestore a run prior to operation as it may be time consuming to read in large quantities of data on line. The output of a run consists of a listing, on the system output tape (A3), and of a BCD tape of graphical data for the CALCOMP plotter, on tape (A6). Tape (A6) contains one file per data case; each file on A6 contains three plots. When requesting CALCOMP outputs, it is necessary to remember how many cases were run, as each case occupies one file.

##### B. Deck

The composition of an AMPSDE deck, ready for operation, is illustrated in Fig. 7. After the usual FORTRAN System Cards, a binary deck consisting of AMPSDE and the CALCOMP Subroutines follows: A FORTRAN System ---\* DATA-- card

---

<sup>†</sup> The program can be used without a plotter, but a buffer tape A6 must then be used.

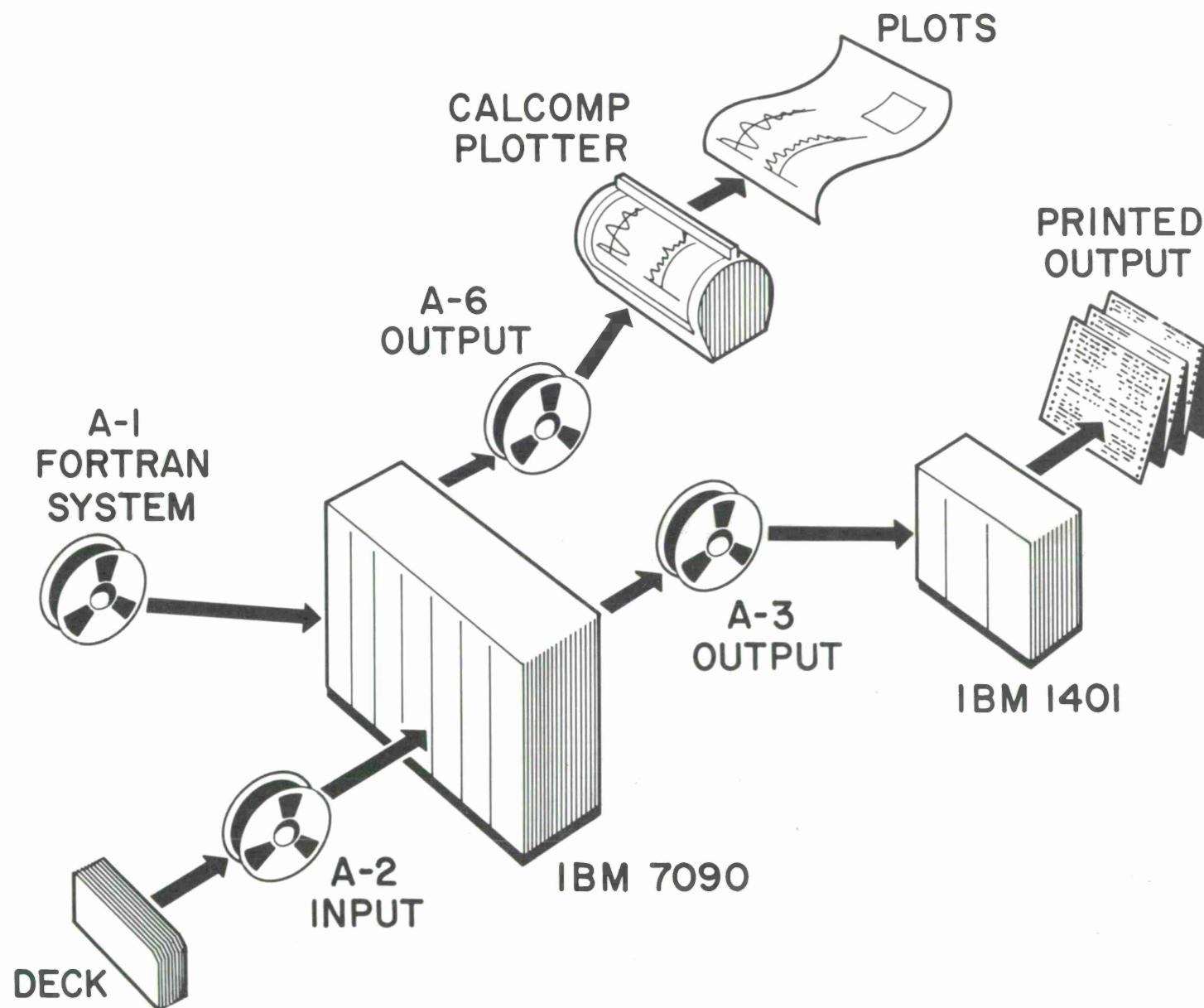
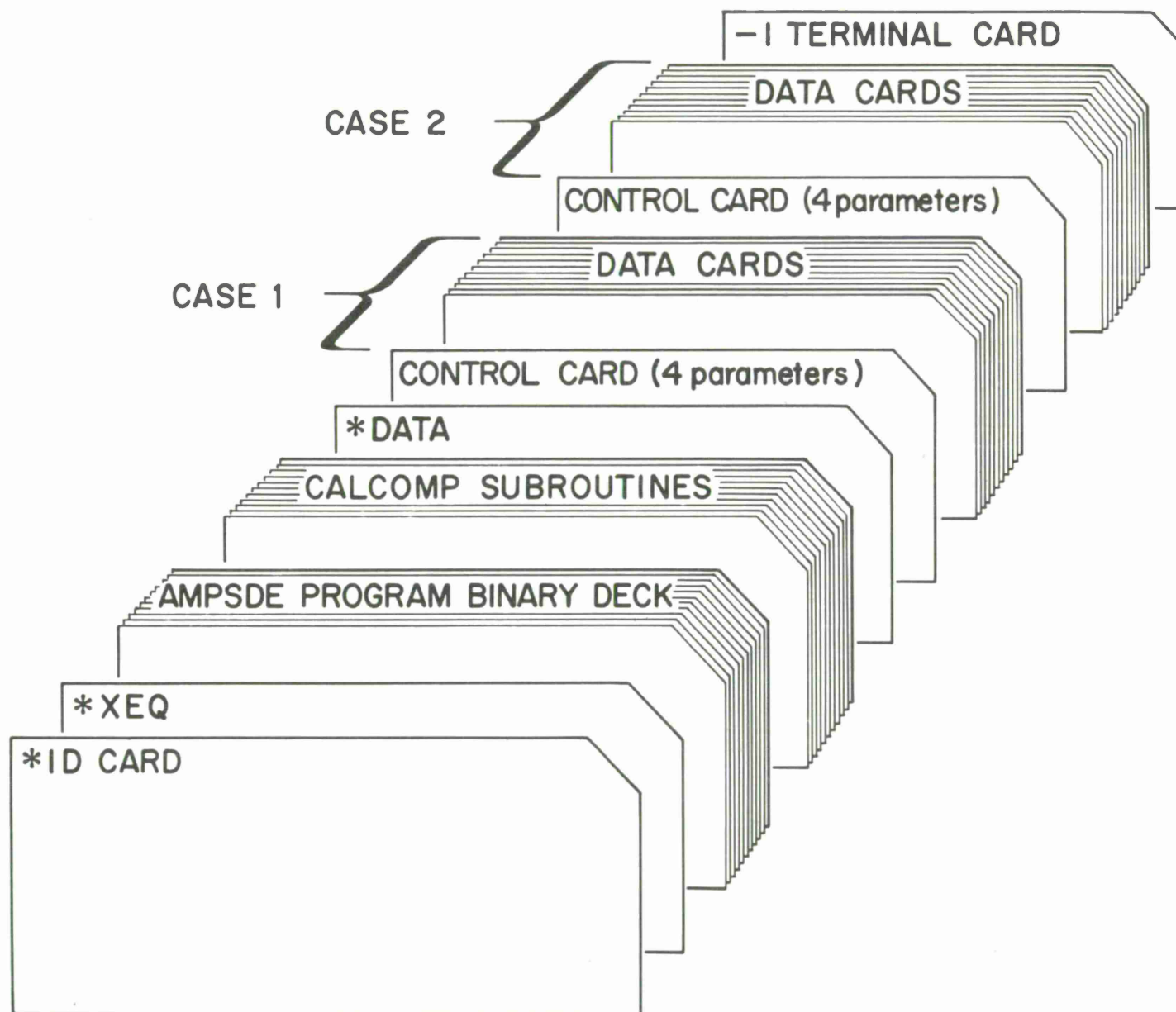


Fig. 6 Usage Diagram



c28-220

Fig. 7 Deck Composition

precedes the data which may consist of several cases. Each case is made up of one control card and a deck of data cards. The last card of the composite deck should contain a negative fixed point integer in the first field (I10), signaling to the main program that all cases have been completed.

#### C. Card Formats

There are three types of cards to describe. These are:

(i) Control Cards: This card contains the four control parameters of the case in the format:

Format	I 10	I 10	F 10.8	F 10.5
Variable	N	K	DT	SF

N+1 is the number of data points

K is the maximum lag index in calculating autocovariances

DT is the data spacing

SF is a scale factor governing the scale of the plots.

(ii) Data Card: Each card contains 7 floating point values in Column 1-70 format F 10.7.

(iii) End of Run Card: This card signals that all cases in a given run have been processed. It contains a negative fixed point integer in the first I 10 field.

#### D. Outputs

Sample output plots have been described in Section III.C. For completeness we shall give a full list of the programs output. The numbering corresponds to the circled numbers on the flow chart in Fig. 1. The first six outputs are printed outputs. The last three outputs are graphical.

(i) The four control parameters are printed as

N = \_\_\_\_\_  
 K = \_\_\_\_\_  
 DT = \_\_\_\_\_  
 SF = \_\_\_\_\_

- (ii) The first 200 values of input data are listed sequentially, 8 per line.
- (iii) The mean and variance of the data are given in the following form:  
 THE MEAN OR DC COMPONENT OF THE INPUT = \_\_\_\_\_  
 THE VARIANCE OF THE INPUT = \_\_\_\_\_
- (iv) The normalized sample covariances are listed 8 per line, starting with the normalized variance (autocorrelation with lag 0), which is identically equal to 1.
- (v) The estimated spectral density is listed, 8 values per line.
- (vi) The power levels in db. corresponding to the estimated spectral density are listed. These are the values plotted on the log-log plot.
- (vii) A plot of the normalized sample autocovariances (autocorrelations).
- (viii) A plot of the estimated Power Spectral Density (PSD).
- (ix) A log-log plot of the PSD.

A complete sample printed output (i to vi) is included as Appendix C.

#### E. Limitations

The maximum number of data points that the program can handle is 10,000. The maximum lag index that the program can accept is 1000. There are no other restrictions apart from those imposed by Format Statements, e.g., input data is assumed to be of the form F 10.7. A trivial modification will permit values larger than 1000 for K to be accepted by the program.

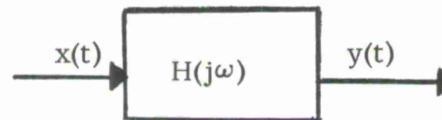
#### V. APPLICATION: ESTIMATION OF TRANSFER FUNCTION GAIN CHARACTERISTICS

In this section, we describe the use of the AMPSDE program in the estimation of the transfer function of both real and simulated systems. When we say transfer function, a linear system is usually implied, but the program may be used just as readily to measure the spectral transfer characteristics of nonlinear systems.



### A. Linear Systems

As is well-known, a single-input/single-output linear system is completely characterized by its impulse response or its transfer function  $H(j\omega)$ . In general  $H(j\omega)$  is a complex valued function and thus it is often convenient to present the gain (i. e. modulus  $|H(j\omega)|$ ) and phase ( $\phi = \arg H(j\omega)$ ) separately as functions of frequency. It is



customary in engineering practice to present these plots in a nonparametric form called the Bode diagram (Ref.[9]) wherein the gain characteristic is plotted in decibels (db.) versus a logarithmic frequency scale, and the phase is plotted in degrees versus log frequency.

In many applications (e. g., radar and sonar systems) the phase  $\phi$  is not important and thus the principal concern is the gain characteristic  $|H(j\omega)|$ . It is to the determination of this characteristic which the present program is particularly well suited. If, however, the phase information is also required, it may be estimated independently via cross-correlation techniques.

For the application at hand, the utility of the AMPSDE program hinges upon the relation between the input/output spectral densities for linear systems; namely,<sup>†</sup>

$$S_y(f) = |H(j\omega)|^2 S_x(f) \quad \omega = 2\pi f$$

where  $S_x(f)$  is the spectral density of the input and  $S_y(f)$  is the output spectral density. Solving for  $\log |H(j\omega)|$ , we have

$$2 \log |H(j\omega)| = \log S_y(f) - \log S_x(f)$$

---

<sup>†</sup> Cf. Ref. [10], Eq. 9-36

or better, in db's,

$$20 \log |H(j\omega)| = 10 \log S_y(f) - 10 \log S_x(f) ,$$

and thus we may obtain the gain characteristic as the difference of two log spectral estimates.

In the application of the program to the determination of Bode gain characteristic for a real physical system, one excites the system with some broad-band excitation (noise or possibly an impulse) and records both the input and output of the system. The data may be taken directly in digital form or as analogue tape recordings which are subsequently converted to digital form by an analog-to-digital (A/D) converter system. In either case, the program is then used to estimate the spectra of both the input and output record and the gain characteristic is obtained by differencing the log-log estimates. This final differencing is not done by the program as it might not be necessary to calculate both estimates. This is especially true for those applications in which the excitation signal is preserved and reused in many spectral determinations. Moreover, (cf., next paragraph), since we are free to choose the input  $x(t)$  in the study of simulated systems, it may well be that  $S_x(f)$  is known and need not be estimated.

One of the principal applications of the program to date by the authors has been the analysis of simulated systems and the evaluation of various digital filtering schemes. In these applications, the simulated system or filter is excited by a known sequence (deterministic or random) for which the spectrum  $S_x(f)$  is known or precalculated. Two very important excitation functions in system studies are the impulse function and white noise, both of which have flat spectra (i.e.,  $S_x(f) = \text{constant}$ ). In these cases, the Bode plot is determined to within a factor by the single estimation of the simulation's output spectra. The program has been very successfully used in this capacity to evaluate some simple, yet, very efficient recursive schemes for digital filtering. The real advantage of the technique is that it allows one to measure the numerical (e.g., round-off) errors and the stability of a filtering scheme for which a direct error and stability analysis may be very difficult.

As an example of the application of the program to the estimation of Bode gain characteristic, consider the simple three element linear recursive scheme

$$y(n) = Ax(n) + By(n-1) + Cy(n-2)$$

which is a discrete version corresponding to the continuous 2nd order system

$$H(s) = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2}$$

For the particular choice of

$$A = 0.00872 \quad B = 1.97330 \quad C = -0.98202$$

which is the sampled continuous system corresponds to

$$\omega_0 = 6\pi \quad \xi = 0.05 \quad \Delta t = 0.005$$

the gain characteristic has been calculated from both the impulse and noise response. The results are presented in Figs. 8 and 9. Figure 10 gives the spectrum of the input noise sample used to drive the recursive scheme. The noise sample was the same one as was used in the estimate given in Fig. 4, but here the resolution in frequency is doubled ( $K = 400$  as opposed to  $K = 200$ ). By rights, the estimated gain characteristic is the difference of the curves of Figs. 9 and 10. However, since the overall input spectrum is flat, the small local variations in the output spectrum (Fig. 9) do not detract much from a meaningful understanding of the gain characteristic.

#### B. Nonlinear Systems

For nonlinear systems, the principle of superposition does not hold and the spectral decompositions of a system's input and output are not in one-to-one correspondence. However, many system studies do require measures of signal distortion and other spectral changes made by a system. The present AMPSDE program is also an efficient tool in this area. As an example, we give in Fig. 11 the output spectrum of a half-wave rectifier under a ten cycle sine wave excitation. We leave it to the reader to verify for himself that the resulting spectral components are the correct one and have the correct relative magnitude.

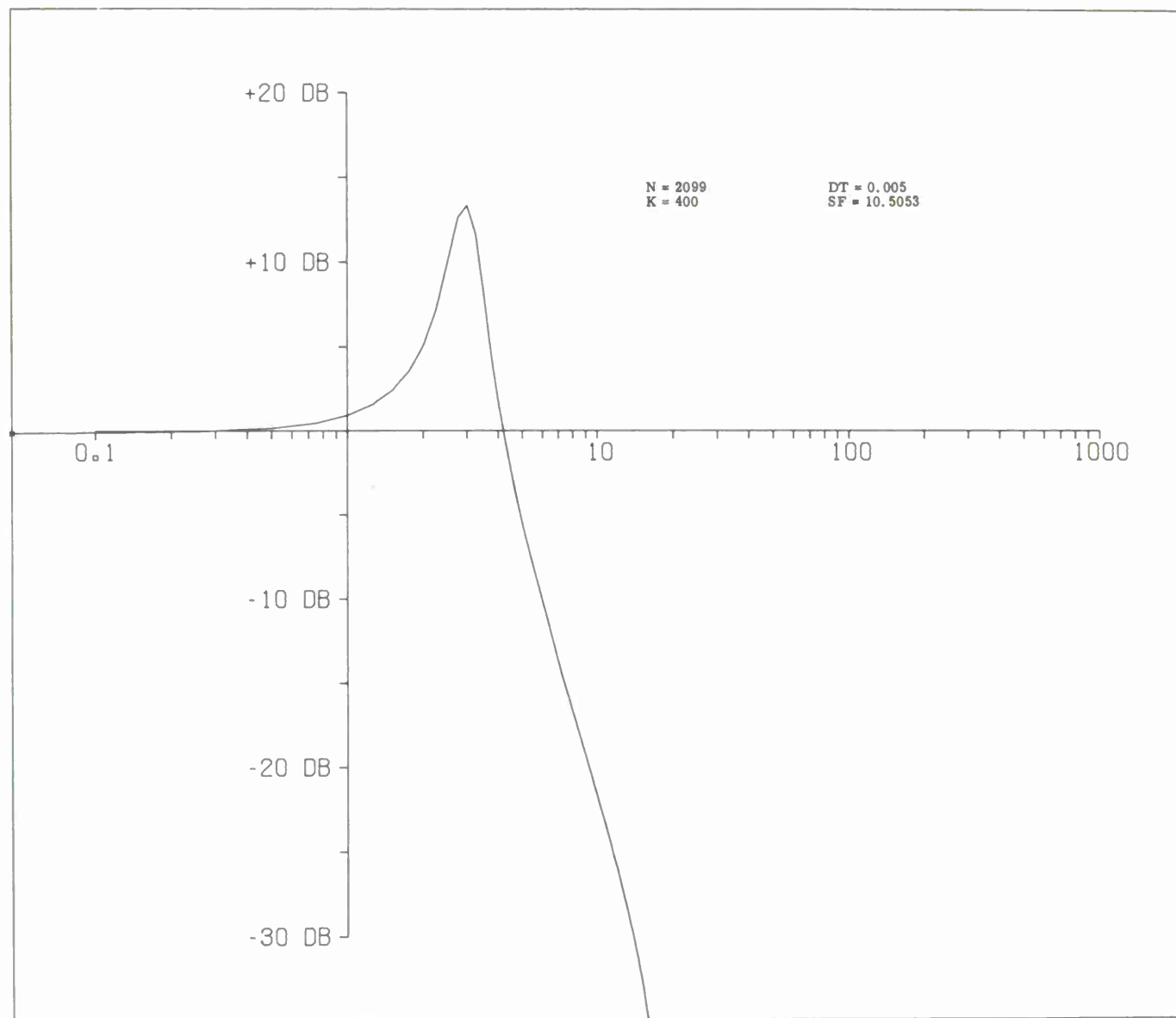


Fig. 8 Frequency Response of Difference Equation from a Unit Impulse

c28-229

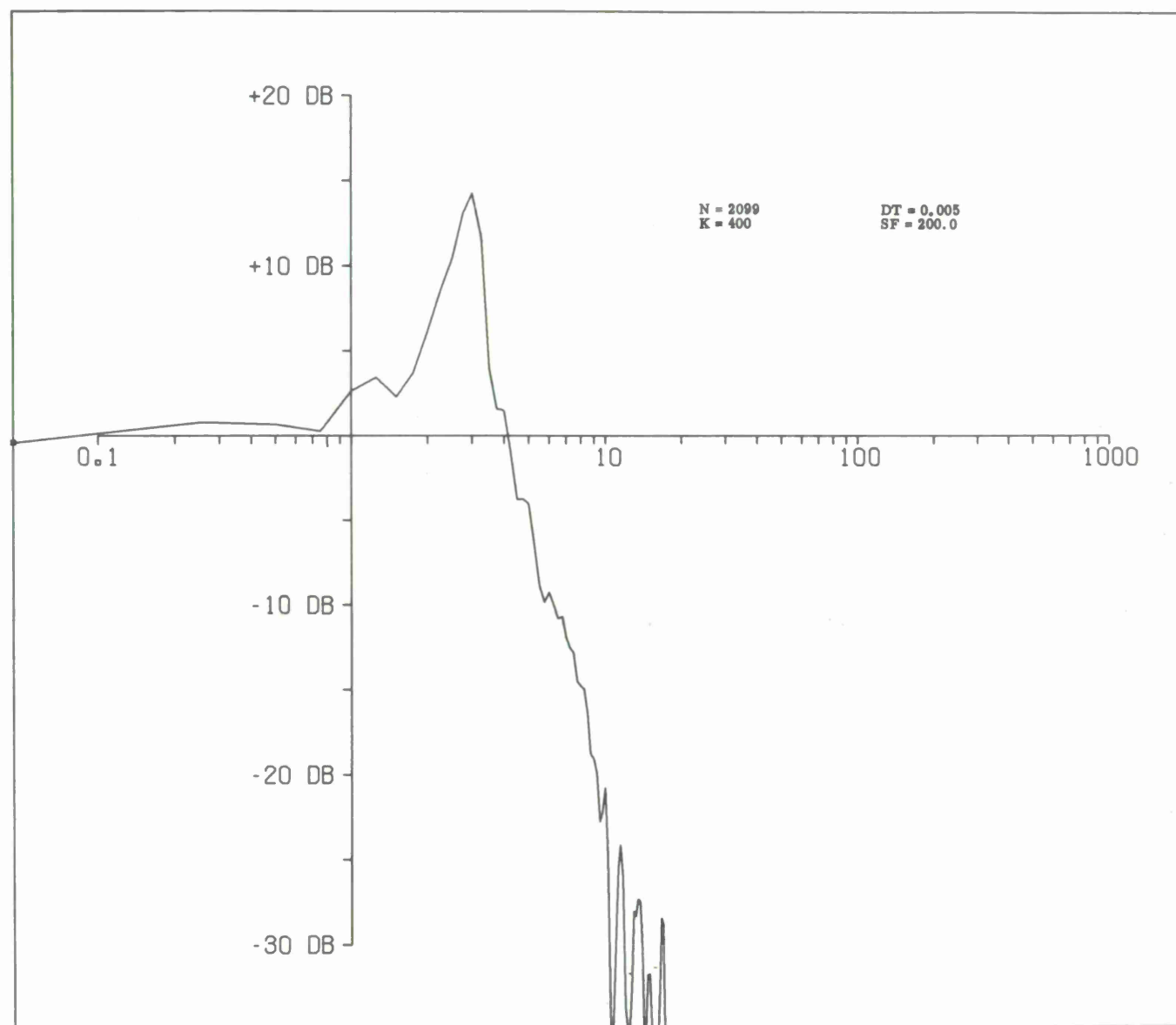


Fig. 9 Frequency Response of Difference Equation from White Gaussian Noise

c28-231

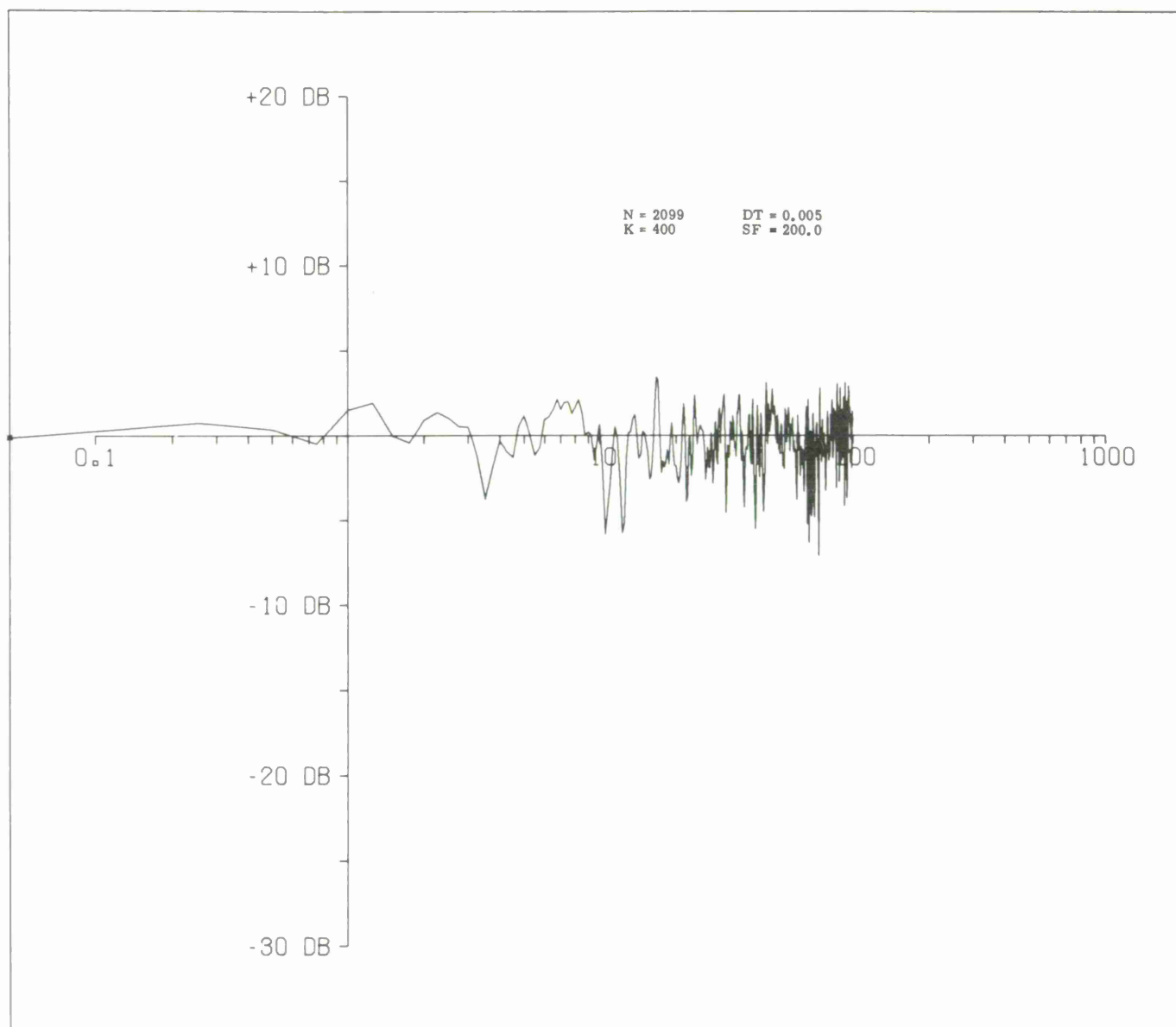
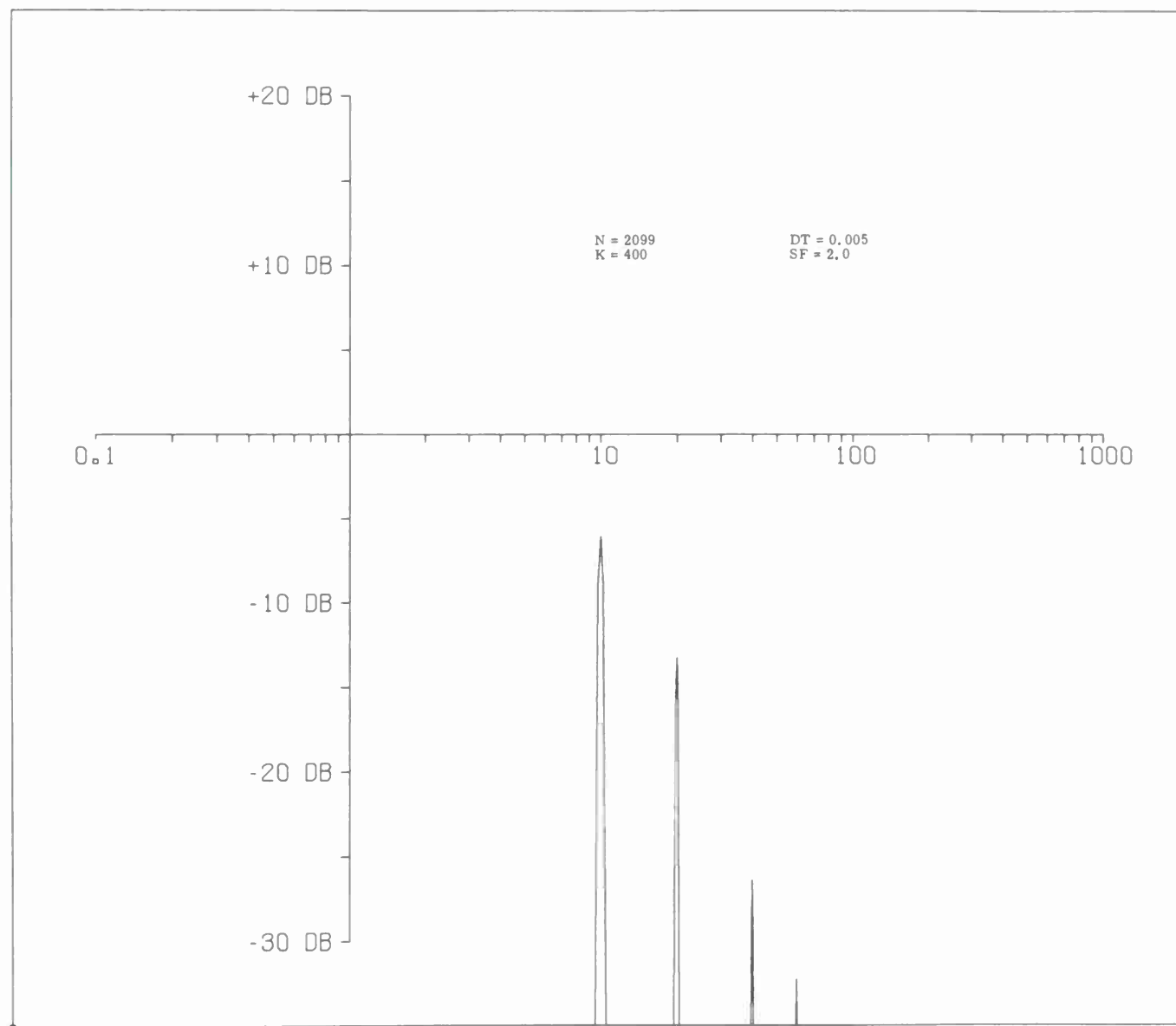


Fig. 10 Spectra of White Gaussian Noise

c28-233



c28-230

Fig. 11 Output Spectra of Half-wave Rectifier Under Sine Wave Excitation

## VI. GUIDELINES TO POSSIBLE MODIFICATIONS

The four principal areas in which modifications of the program might be indicated are:

- Input
- Autocovariance estimation
- Smoothing of the periodogram
- Output

### A. Input

It has already been noted that input is made via punched cards, and that prestoring a run obviates the objection of slow read in. If, however, data is available on tape in either BCD or binary form, it is easy to modify the input portion of the program to handle the new medium.

### B. Calculation of Sample Autocovariance

The program estimates the autocovariances of the process by the sample covariances using the formula

$$C = \frac{1}{N - |k|} \sum_{n=0}^{N-|k|} x_n x_{n+|k|} , \quad k = 0, 1, \dots, K.$$

Several authors prefer to use

$$C_k = \frac{1}{N} \sum_{n=0}^{N-|k|} x_n x_{n+|k|} .$$

Other modifications are possible, amounting to weighting suitably the autocovariances, before taking the cosine transform (cf. Ref. [11], p. 59 ff.). If one or another of the covariance estimates appears preferable to the estimate used in the program, it is easy to replace the section of the program headed "CALCULATE AUTOCORRELATION FUNCTION" with an alternative.



### C. Periodogram Smoothing

The program section which performs the smoothing of the periodogram consists of precisely five statements. These can readily be replaced by another smoothing sequence. Note that to each periodogram smoothing sequence, there corresponds an autocorrelation weighting scheme and vice versa. It appears, however, simpler in digital processing to do the smoothing after the cosine transform has been taken, i.e., in the frequency domain, rather than in the lag domain.

### D. Output

At user's option, minor modifications will insure the omission of any of the presently provided outputs. Additional outputs, however, will entail more work, as required.

APPENDIX A  
FINITE FOURIER COSINE TRANSFORMS

The Fourier Cosine Transform of a function  $f(t)$  is defined as

$$(A1) \quad A(\omega) = \int_0^{\infty} f(t) \cos \omega t \, dt.$$

The corresponding inversion formula

$$(A2) \quad f(t) = \frac{1}{\pi} \int_0^{\infty} A(\omega) \cos \omega t \, d\omega$$

also holds. If, instead of being defined on the whole axis,  $f(t)$  is given at only a finite number of equi-spaced points

$$(A3) \quad f_0 = f(0), f_1 = f(\Delta t), f_2 = f(2\Delta t), \dots, f_N = f(N\Delta t),$$

the Fourier transform can be naturally approximated by a numerical integration scheme such as the trapezoidal rule. Equation (A1) then becomes

$$(A4) \quad A(\omega) = \left\{ \frac{1}{2} f_0 + \sum_{j=1}^{N-1} f_j \cos(j\omega\Delta t) + \frac{1}{2} f_N \cos N\omega\Delta t \right\} \Delta t.$$

According to the sampling theorem, the meaningful range of  $\omega$  is the interval  $(0, \frac{\pi}{\Delta t})$ , and since  $N+1$  values of  $f$  are available we can select  $N+1$  values of  $\omega$  according to

$$(A5) \quad \omega_k = \frac{\pi}{N\Delta t}, \quad k = 0, 1, 2, \dots, N.$$

The discrete analog of (A4) now becomes

$$(A6) \quad A_k = A(\omega_k) = \left\{ \frac{1}{2} f_0 + \sum_{j=1}^{N-1} f_j \cos \left( \frac{jk\pi}{N} \right) + \frac{1}{2} (-1)^k f_N \right\} \Delta t$$

Note that an ordinary Fourier cosine expansion of the sample  $\{f_j\}$  yields

$$(A7) \quad a_k = \frac{2}{N\Delta t} A_k = \frac{2}{N} \left\{ \frac{1}{2} f_0 + \sum_{j=1}^{N-1} f_j \cos \left( \frac{jk\pi}{N} \right) + \frac{1}{2} (-1)^k f_N \right\}$$

for the  $k^{\text{th}}$  Fourier Cosine Coefficient of  $f$ . Thus, if we define

$$(A8) \quad \tilde{A}_k = \frac{1}{2} f_0 + \sum_{j=1}^{N-1} f_j \cos \frac{jk\pi}{N} + \frac{1}{2} (-1)^k f_N,$$

the ordinary Fourier coefficient is given by

$$(A9) \quad a_k = \frac{2}{N} \tilde{A}_k$$

and the discrete Fourier cosine transform of  $f$  is given by

$$(A10) \quad A_k = \Delta t \tilde{A}_k.$$

## APPENDIX B

### PROGRAM LISTING

AMPSDE \*\* ANY MAN'S POWER SPECTRAL DENSITY ESTIMATOR \*\* C R ARNOLD 12/17/64

```

C
C
C
C      DATA CARD 1 = CONTROL PARAMETERS
C      COL. 1-10 = N (FIXED INTEGER = NO. OF DATA VALUES MINUS ONE)
C      COL. 11-20 = K (FIXED INTEGER = MAXIMUM LAG INDEX)
C      COL. 21-30 = DT (FLT. PT. NO. = TIME INCREMENT BETWEEN DATA POINTS)
C      COL. 31-40 = SF (FLT. PT. NO. = SCALE FACTOR FOR SPECTRAL DENSITY)
C
C      DATA CARDS 2- = FLT. PT. DATA VALUES, 7 PER CARD, COLS. 1-70
C
C      PLOTS OF THE NORMALIZED AUTOCORRELATION FUNCTION, AND THE POWER SPECTRAL
C      DENSITY (ONE HALF OF TWO SIDED SPECTRA) ON BOTH A LINEAR AND A LOG-LOG
C      SCALE ARE GENERATED ON TAPE A6 WITH ONE FILE FOR EACH SET OF DATA
C
C      PROGRAM WILL TAKE MULTIPLE SETS OF DATA/EACH SET HAVING A CONTROL CARD
C
C      PROGRAM NORMALLY TERMINATED WITH A CONTROL CARD WITH A NEGATIVE
C      INTEGER IN THE FIRST 10 COLUMNS
C
C
C      DIMENSION X(10005),R(1001),C(2001),V(1001),P(1001),BUFFER(1024)
C      CALL PLOTS(BUFFER(1024),1024)
C
C      CALL PLOT(24.0,-24.0,-3)
C      CALL PLOT(0.0,16.0,-3)
C
C
C
C      READ IN CONTROL PARAMETERS
C
C      1 READ INPUT TAPE 2,2,NN,KK,DT,SF
C      2 FORMAT(2I10,F10.8,F10.5)
C      IF(NN)150,150,3
C      3 WRITE OUTPUT TAPE 3,4,NN,KK,DT,SF
C      4 FORMAT(1H1,8X,42HANY MAN'S POWER SPECTRAL DENSITY ESTIMATOR,3X,2H*
C      1*,3X,10HC R ARNOLD,3X,2H*,3X,8HGROUP 28,///9X,24HINPUT CONTROL P
C      2ARAMETERS,34X,18H LINCOLN LABORATORY,///9X,3HN =,15,///18X,3HK =,I
C      34,///27X,4HDT =,F8.5,///36X,4HSF =,F10.5,////)
C
C
C      CALCULATE CONSTANTS
C
C      NP1=NN+1
C      NP2=NN+2
C      NP3=NN+3
C      KP1=KK+1
C      KP2=KK+2
C      FK=KK
C      FNP1=NP1
C      DELTA=0.05
C      FMM=0.43429448
C      CP =2.0*FMM

```

## APPENDIX B

AMPSCE \*\* ANY MAN'S POWER SPECTRAL DENSITY ESTIMATOR \*\* C R ARNOLD 12/17/64

```

      CF =3.0*FMM
C
C
C      READ IN DATA
C
      READ INPUT TAPE 2,6,(X(I),I=1,NP1)
      6 FORMAT(7F10.7)
      WRITE OUTPUT TAPE 3,8
      8 FORMAT(1H1,8X,30HFIRST 200 VALUES OF INPUT DATA,/)
      WRITE OUTPUT TAPE 3,10,(X(I),I=1,200)
      10 FORMAT(1H ,8F15.6)
C
C
C      CALCULATE AND REMOVE DC COMPONENT OF THE DATA
C
      SUM=0.0
      DO 12 I=1,NP1
      SUM=SUM+X(I)
      12 CONTINUE
      CC=SUM/FNP1
      DO 14 I=1,NP1
      X(I)=X(I)-DC
      14 CONTINUE
      WRITE OUTPUT TAPE 3,16,DC
      16 FORMAT(1H1,8X,39HTHE MEAN OR DC COMPONENT OF THE INPUT =,F9.6,/)
C
C
C      CALCULATE AUTO-CORRELATION FUNCTION
C
      DO 20 K=1,KP1
      SUM=0.0
      LIM=NP2-K
      DO 18 I=1,LIM
      KX=I+K-1
      SUM=SUM+X(I)*X(KX)
      18 CONTINUE
      FLIM=LIM
      R(K)=SUM/FLIM
      20 CONTINUE
      VAR=R(1)
      WRITE OUTPUT TAPE 3,22,VAR
      22 FORMAT(1H0,///1H0,///1H0,8X,27HTHE VARIANCE OF THE INPUT =,F12.8,/)
C
C
C      CALCULATE AND PRINT NORMALIZED AUTO-CORRELATION
C
      DO 24 K=1,KP1
      R(K)=R(K)/VAR
      24 CONTINUE
      WRITE OUTPUT TAPE 3,26
      26 FORMAT(1H1,8X,31HTHE NORMALIZED AUTO-CORRELATION,/)
      WRITE OUTPUT TAPE 3,28,(R(K),K=1,KP1)
      28 FORMAT(1H ,8F15.6)
C
C
C      GENERATE TABLE OF COSINE VALUES

```

## APPENDIX B

AMPSDE \*\* ANY MAN'S POWER SPECTRAL DENSITY ESTIMATOR \*\* C R ARNOLD 12/17/64

```

      KB=2*KP1
      CNST1=3.14159265/FK
      DO 40 I=1,KP1
      ALPHA=I-1
      C(I)=COSF(ALPHA*CNST1)
40    CONTINUE
      DO 45 I=1,KK
      KX=KB-I
      C(KX)=C(I)
45    CONTINUE

C
C      CALCULATE COSINE TRANSFORM
C
      KX=0
      R(1)=0.5*R(1)
      R(KP1)=0.5*R(KP1)
      MOD=2*KK+1
50    KB=1
      DO 55 I=1,KP1
      V(I)=C(KB)
      KB=KB+KX
      IF(MOD-KB)52,55,55
52    KB=KB+1-MOD
55    CONTINUE
      SUM=0.0
      DO 60 I=1,KP1
      SUM=SUM+V(I)*R(I)
60    CONTINUE
      JX=KX+1
      P(JX)=2.0*DT*SUM
      IF(KX-KK)62,65,65
62    KX=KX+1
      GO TO 50
65    R(1)=2.0*R(1)
      R(KP1)=2.0*R(KP1)

C
C      SMOOTH SPECTRAL DENSITY WITH HANNING WEIGHTS
C
      V(1)=0.5*P(1)+0.5*P(2)
      DO 66 I=2,KK
      V(I)=0.25*P(I-1)+0.5*P(I)+0.25*P(I+1)
66    CONTINUE
      V(KP1)=0.5*P(KK)+0.5*P(KP1)

C
C      APPLY VARIANCE OR SCALE FACTOR TO ESTIMATE
C
      IF(SF)67,67,68
67    SF=1.0/DT
      GO TO 69

C
68    SF=SF*VAR
69    DO 70 I=1,KP1
      V(I)=SF*V(I)
      P(I)=V(I)
70    CONTINUE

```

## APPENDIX B

AMPSCE \*\* ANY MAN'S POWER SPECTRAL DENSITY ESTIMATOR \*\* C R ARNOLD 12/17/64

```

C
C
C      PRINT SPECTRAL ESTIMATE
C
C      WRITE OUTPUT TAPE 3,72
72  FORMAT(1H1,8X,30HTHE ESTIMATED SPECTRAL DENSITY,/)
C      WRITE OUTPUT TAPE 3,28,(P(I),I=1,KP1)
C
C
C
C
C      GENERATE TIME SCALE FOR CALCOMP PLOT OF AUTO-CORRELATION
C
C      TMIN=0.0
C      SK=KK
C      TMAX=SK*DT
C      TINC=200.0*CT
C      SK=SK/20.0
C      SKIP=SK+4.0
C
C
C
C
C      SCALE AND PLOT NORMALIZED AUTO-CORRELATION
C
C      R(KP2)=-1.0
C      CALL SCALE(R,KP2,10.0,RMIN,DR)
C      CALL AXIS(0.0,0.0,15HAUTOCORRELATION,15,10.0,90.0,RMIN,DR)
C      CALL AXIS(0.0,5.0,3HLAG,3,SK,0.0,TMIN,TINC)
C      I3=3
C      T=0.0
C      DO 30 I=1,KP1
C      CALL PLOT(T,R(I),I3)
C      I3=2
C      T=T+DELTA
30  CONTINUE
C
C
C      CALL PLOT(0.0,-13.0,-3)
C
C
C
C
C      GENERATE FREQUENCY SCALE FOR CALCOMP PLOT OF SPECTRAL DENSITY
C
C      QMIN=0.0
C      QMAX=0.5/DT
C      CQ=100.0/TMAX
C
C
C
C
C      SCALE AND PLOT ESTIMATED SPECTRAL DENSITY
C
C      V(KP2)=0.0
C      CALL SCALE(V,KP2,10.0,PMIN,DP)
C      CALL AXIS(0.0,0.0,13HPOWER DENSITY,13,10.0,90.0,PMIN,DP)

```

## APPENDIX B

AMPSDE \*\* ANY MAN'S POWER SPECTRAL DENSITY ESTIMATOR \*\* C R ARNOLD 12/17/64

```

      CALL AXIS(0.0,0.0,9HFREQUENCY,9,SK,0.0,QMIN,DQ)
      I3=3
      T=0.0
      DO 75 I=1,KP1
      CALL PLOT(T,V(I),I3)
      I3=2
      T=T+DELTA
75 CONTINUE

C
C
C      CALCULATE POWER LEVELS IN DB (IE. LOG SCALE)
C
      DO 104 I=1,KP1
      IF(P(I)-0.00031623)102,103,103
102 P(I)=0.00031623
103 P(I)=CP*LOGF(P(I))
      V(I)=5.0*P(I)
104 CONTINUE
      WRITE OUTPUT TAPE 3,105
105 FORMAT(1H1,8X,18HPOWER LEVELS IN DB,///)
      WRITE OUTPUT TAPE 3,28,(V(I),I=1,KP1)

C
C
C      CALCULATE LOG(FREQ) VALUES
C
      DF=1.0/(2.0*DT*FK)
      V(1)=0.0
      DO 106 I=2,KP1
      V(I)=V(I-1)+DF
106 CONTINUE
      V(1)=-4.0
      DO 108 I=2,KP1
      V(I)=CF*LOGF(V(I))
108 CONTINUE

C
C
C      CALL PLOT(SKIP,0.0,-3)

C
C
C      CALL PLOT(14.0,0.0,2)
      CALL PLOT(14.0,12.0,2)
      CALL PLOT(0.0,12.0,2)
      CALL PLOT(0.0,0.0,2)

C
C
C      CALL PLOT(4.0,7.0,-3)

C
C
C      PLOT ESTIMATE ON LOG-LOG PLOT (BODE PLOT)

      CALL SYMBL4(V,P,0.2,1H*,0.0,1)
      I3=3
      DO 110 I=1,KP1
      CALL PLOT(V(I),P(I),I3)
110 I3=2
C

```



## APPENDIX B

AMPSCE \*\* ANY MAN'S POWER SPECTRAL DENSITY ESTIMATOR \*\* C R ARNOLD 12/17/64

```

C      PLCT AND LABEL POWER AXIS
C
      CALL AXIS4(0.0,-6.0,10.0,90.0)
      CALL SYMBL4(-1.20,3.9,0.2,6H+20 DB,0.0,6)
      CALL SYMBL4(-1.20,1.9,0.2,6H+10 DB,0.0,6)
      CALL SYMBL4(-1.2,-2.1,0.2,6H-10 DB,0.0,6)
      CALL SYMBL4(-1.2,-4.1,0.2,6H-20 DB,0.0,6)
      CALL SYMBL4(-1.2,-6.1,0.2,6H-30 DB,0.0,6)
C
C      PLCT AND LABEL FREQUENCY AXIS
C
      V(1)=0.1
      CC 112 I=2,10
      V(I)=V(I-1)+0.1
112  CCNTINUE
      CC 114 I=11,19
      V(I)=V(I-1)+1.0
114  CCNTINUE
      CC 116 I=20,28
      V(I)=V(I-1)+10.0
116  CCNTINUE
      CC 118 I=29,37
      V(I)=V(I-1)+100.0
118  CCNTINUE
      CC 120 I=1,37
      V(I)=CF*LCGF(V(I))
120  CCNTINUE
      CALL PLOT(V(1),-0.1,3)
      CC 130 I=1,36
      CALL PLCT(V(I),0.0,2)
      XXX=V(I+1)
      CALL PLOT(XXX,0.0,2)
      CALL PLOT(XXX,-0.1,2)
130  CCNTINUE
      CALL SYMBL4(+0.70,-0.35,0.2,4H1000,0.0,4)
      CALL SYMBL4(+5.80,-0.35,0.2,3H100,0.0,3)
      CALL SYMBL4(+2.90,-0.35,0.2,2H10,0.0,2)
      CALL SYMBL4(-3.23,-0.35,0.2,3H0.1,0.0,3)
C
C      END PLOT AND GO BACK FOR NEXT SAMPLE
C
140  CALL PLOT(20.0,6.0,-3)
      CALL PLOT(0.0,0.0,-3)
      END FILE 6
      GO TC 1
C
C      END PROGRAM
C
150  END FILE 6
      END FILE 6
      END FILE 6
      CALL RUN(1,6)
      CALL EXIT
      ENC(1,1,0,0,0,0,1,1,0,0,0,0,0,0,0)

```

APPENDIX C  
SAMPLED LISTED OUTPUTS

ANY MAN'S POWER SPECTRAL DENSITY ESTIMATOR \*\* C R ARNOLD \*\* GROUP 28

INPUT CONTROL PARAMETERS

LINCOLN LABORATORY

N = 2099

K = 200

DT = 0.00500

SF = 0.

# APPENDIX C

## FIRST 200 VALUES OF INPUT DATA

0.003167	-0.005749	-0.011367	-0.005163	-0.009799	-0.003509	-0.008939	-0.019075
-0.023012	-0.017448	0.001370	0.000953	-0.002311	-0.002168	-0.004476	-0.006565
0.001821	0.005840	0.001003	0.002131	-0.002987	-0.014390	-0.014641	-0.012506
-0.015544	-0.012399	-0.014717	-0.029699	-0.022186	-0.027522	-0.030688	-0.022335
-0.005086	-0.010975	-0.019002	-0.011417	-0.012012	-0.012035	-0.022996	-0.024289
-0.005054	-0.010342	-0.010682	-0.002371	-0.003004	-0.012269	-0.013185	-0.008765
-0.027427	-0.026475	-0.019039	-0.020300	-0.026474	-0.024630	-0.025687	-0.034429
-0.017369	-0.014409	-0.007360	-0.002180	-0.004654	-0.005970	-0.013058	-0.020007
-0.023924	-0.012422	-0.006821	-0.003172	0.004044	0.002672	0.000501	-0.001307
0.000895	-0.002061	-0.000947	0.001386	0.005832	-0.005780	-0.006116	-0.002693
-0.003641	0.000937	0.005074	0.018447	0.012427	0.007912	0.004069	0.013192
0.013763	0.011450	0.010923	0.010053	0.000033	0.015801	0.011591	0.015092
0.006324	0.006650	0.007713	0.009340	0.012714	0.031227	0.036553	0.042212
0.043818	0.040871	0.045449	0.051783	0.039775	0.045987	0.041747	0.050237
0.059757	0.053769	0.045771	0.056794	0.047739	0.054025	0.059218	0.050128
0.056577	0.053083	0.038207	0.049293	0.052675	0.055173	0.056187	0.058741
0.061506	0.070521	0.071150	0.053271	0.050443	0.049653	0.042964	0.038961
0.022936	0.021861	0.022808	0.031506	0.033402	0.042664	0.041065	0.038351
0.035334	0.038199	0.047659	0.045418	0.043687	0.041664	0.042470	0.047433
0.057114	0.046289	0.042176	0.048073	0.041371	0.030891	0.029886	0.028271
0.022063	0.023262	0.029863	0.027031	0.032825	0.030824	0.035899	0.038218
0.029074	0.033535	0.028399	0.035968	0.022993	0.020388	0.021802	0.012861
0.019606	0.011882	0.008029	0.004099	-0.000928	0.000658	-0.008010	-0.016752
-0.023026	-0.009547	-0.000501	0.006225	0.008641	0.006158	0.006963	0.005466
0.002031	-0.002342	-0.009302	0.002381	-0.013952	-0.011380	-0.013439	-0.017056

## APPENDIX C

THE MEAN OR DC COMPONENT OF THE INPUT =  $0.001677$

THE VARIANCE OF THE INPUT =  $0.00221946$

# APPENDIX C

## THE NORMALIZED AUTO-CORRELATION

1.000000	0.988483	0.977261	0.965370	0.954276	0.942217	0.929966	0.918140
0.905797	0.892993	0.880604	0.868973	0.857512	0.845596	0.833682	0.822193
0.811447	0.801002	0.790465	0.780374	0.770846	0.761966	0.752618	0.743173
0.734260	0.724607	0.713827	0.702710	0.691172	0.679679	0.667647	0.655792
0.644426	0.633210	0.621566	0.609688	0.597800	0.586348	0.575479	0.565433
0.555399	0.545372	0.535604	0.526118	0.516743	0.508154	0.499441	0.490440
0.481260	0.472169	0.463258	0.454627	0.445892	0.437144	0.428739	0.420930
0.413395	0.405310	0.397709	0.389483	0.381805	0.374034	0.366265	0.358213
0.351069	0.344606	0.338576	0.332120	0.326226	0.320630	0.315612	0.310338
0.305072	0.299781	0.293685	0.287475	0.281205	0.275424	0.270487	0.265387
0.260212	0.255367	0.250505	0.246645	0.241424	0.235989	0.229746	0.223451
0.217503	0.211944	0.206554	0.201276	0.196585	0.190545	0.185269	0.179680
0.173913	0.168242	0.162913	0.158431	0.154666	0.150838	0.147311	0.144005
0.140643	0.136972	0.133041	0.128752	0.125338	0.121600	0.117824	0.114668
0.111254	0.108785	0.107270	0.106156	0.104814	0.103885	0.103440	0.102465
0.100383	0.097775	0.095652	0.093491	0.091450	0.088514	0.086227	0.084227
0.082885	0.080207	0.080081	0.079162	0.078252	0.077229	0.075797	0.074900
0.073443	0.072684	0.072061	0.071322	0.070467	0.070309	0.069132	0.067951
0.065955	0.063710	0.061874	0.060037	0.058710	0.056929	0.055543	0.053550
0.052326	0.050885	0.049350	0.047386	0.044867	0.042581	0.039943	0.037485
0.033733	0.030269	0.026932	0.023246	0.019495	0.015940	0.013107	0.009692
0.006207	0.001968	-0.002705	-0.007311	-0.011726	-0.015393	-0.018276	-0.021291
-0.024798	-0.029347	-0.033794	-0.038043	-0.042253	-0.046977	-0.051878	-0.055684
-0.058950	-0.062460	-0.065813	-0.069140	-0.072810	-0.075921	-0.080768	-0.084992
-0.089030	-0.092410	-0.096080	-0.099332	-0.104045	-0.109162	-0.113847	-0.118653
-0.123105							

# APPENDIX C

## THE ESTIMATED SPECTRAL DENSITY

93.581613	72.606593	36.166934	15.285943	7.339891	4.448726	2.873741	1.478441
1.209590	1.159778	1.067970	0.810181	0.682259	0.732379	0.740237	0.656483
0.559497	0.444835	0.321608	0.251954	0.202903	0.125660	0.128743	0.161164
0.105747	0.075026	0.125775	0.156942	0.131297	0.106723	0.106586	0.085253
0.073263	0.121475	0.136069	0.080033	0.050445	0.055309	0.063480	0.058536
0.040737	0.030593	0.043329	0.055089	0.043175	0.035928	0.040190	0.052992
0.055298	0.043584	0.041222	0.038913	0.031075	0.024761	0.025874	0.028505
0.028214	0.029588	0.030122	0.033064	0.036628	0.039529	0.034146	0.020532
0.019161	0.023863	0.025889	0.026414	0.022617	0.020893	0.027652	0.031167
0.024218	0.016438	0.011352	0.011369	0.015252	0.018363	0.017022	0.014108
0.017845	0.015686	0.007391	0.009969	0.015455	0.012838	0.012544	0.013433
0.007867	0.004957	0.011512	0.018343	0.017637	0.016475	0.015812	0.017177
0.0020158	0.019220	0.016525	0.013963	0.013575	0.013100	0.010667	0.010726
0.011950	0.009531	0.006734	0.008498	0.011610	0.011404	0.011458	0.012473
0.010963	0.010003	0.010729	0.010580	0.008943	0.007991	0.007483	0.005267
0.005464	0.008298	0.008797	0.007196	0.006039	0.006110	0.005953	0.004892
0.004753	0.006637	0.008774	0.008139	0.007234	0.007532	0.005802	0.005917
0.007035	0.004808	0.004427	0.006850	0.005963	0.003903	0.004759	0.006591
0.007847	0.006325	0.003559	0.005693	0.009216	0.007620	0.005949	0.007101
0.007304	0.006327	0.005675	0.004520	0.003878	0.004805	0.006551	0.007397
0.006340	0.005318	0.005234	0.006487	0.008283	0.009058	0.007462	0.005628
0.006503	0.007777	0.007837	0.005883	0.005000	0.007896	0.008132	0.005715
0.004930	0.006223	0.008571	0.007431	0.005986	0.006882	0.006119	0.006530
0.006832	0.004602	0.006495	0.009681	0.008133	0.005146	0.004287	0.005610
0.008237	0.009467	0.008217	0.007272	0.006735	0.005779	0.005465	0.006471
0.007243							

# APPENDIX C

## POWER LEVELS IN DB

19.711905	18.609760	15.583116	11.842922	8.656896	6.482357	4.584476	1.698040
0.826383	0.643749	0.285589	-0.914179	-1.660510	-1.352640	-1.306292	-1.827768
-2.522021	-3.518011	-4.926736	-5.986784	-6.927122	-9.008046	-8.902752	-7.927331
-9.757337	-11.247897	-9.004056	-8.042615	-8.817438	-9.717423	-9.723003	-10.692918
-11.351160	-9.155119	-8.662407	-10.967301	-12.971856	-12.572010	-11.973599	-12.325783
-13.900129	-15.143764	-13.632233	-12.589360	-13.647670	-14.445651	-13.958782	-12.757925
-12.572881	-13.606699	-13.848740	-14.099100	-15.075921	-16.062290	-15.871309	-15.450785
-15.495301	-15.288816	-15.211140	-14.806384	-14.361810	-14.030878	-14.666584	-16.875745
-17.175841	-16.222730	-15.868859	-15.781597	-16.455670	-16.800065	-15.582720	-15.063045
-16.158637	-17.841593	-19.449128	-19.442949	-18.166800	-17.360509	-17.689894	-18.505205
-17.484745	-18.044876	-21.313200	-20.013465	-18.109404	-18.915038	-19.015637	-18.718239
-21.0041811	-23.047373	-19.388446	-17.365269	-17.535735	-17.831777	-18.010159	-17.650464
-16.955487	-17.162549	-17.818516	-18.550284	-18.672472	-18.827414	-19.719727	-19.695793
-19.226228	-20.208486	-21.717168	-20.706888	-19.351708	-19.429600	-19.408992	-19.040171
-19.600570	-19.998612	-19.694548	-19.755076	-20.485268	-20.974099	-21.259058	-22.784139
-22.625077	-20.810076	-20.556669	-21.429288	-22.190133	-22.139465	-22.252278	-23.105397
-23.230194	-21.780471	-20.568165	-20.894531	-21.406152	-21.230669	-22.364450	-22.279319
-21.527414	-23.180190	-23.539086	-21.642945	-22.245610	-24.085538	-23.225269	-21.810688
-21.052745	-21.989533	-24.486281	-22.446638	-20.354578	-21.180660	-22.255781	-21.486724
-21.364614	-21.987853	-22.459995	-23.448793	-24.113971	-23.182712	-21.837173	-21.309591
-21.979179	-22.742128	-22.811721	-21.879473	-20.818088	-20.429798	-21.271517	-22.496240
-21.869076	-21.091760	-21.058426	-22.303816	-23.010666	-21.026017	-20.897881	-22.429556
-23.064732	-22.059978	-20.669545	-21.289626	-22.228683	-21.622881	-22.133362	-21.850578
-21.654593	-23.370324	-21.873975	-20.140699	-20.897345	-22.885211	-23.678906	-22.510258
-20.842142	-20.238010	-20.852751	-21.383698	-21.716848	-22.381790	-22.623821	-21.890021
-21.400802							

## REFERENCES

1. E. Korngold, "The Periodic Analysis of Sampled Data," Lincoln Laboratory Group Report 1964-32 (15 June 1964).
2. G.M. Jenkins, "General Considerations in the Analysis of Spectra," Technometrics, 3 (1961), pp. 133-166.
3. R.B. Blackman and J.W. Tukey, The Measurement of Power Spectra, Dover (1959).
4. G.M. Jenkins and M.B. Priestley, "The Spectral Analysis of Time Series," J.Roy.Stat.Soc., Series B, 19 (1957), pp. 13-37.
5. M.B. Priestley, "Basic Considerations in the Estimation of Spectra," Technometrics, 4(1962), pp. 551-564.
6. E. Parzen, "Mathematical Considerations in the Estimation of Spectra," Technometrics, 3(1961), pp. 167-190.
7. A.K. Susskind, "Sampling and Quantizing," Chapter II in Notes on Analog-Digital Conversion Techniques, ed. A.K. Susskind, John Wiley (1958).
8. California Computer Products, Inc., "CALCOMP Plotter Subroutines for IBM Computers," Bulletin 120B (March 1960).
9. H.W. Bode, Network Analysis and Feedback Amplifier Design, Van Nostrand (1945).
10. W.B. Davenport, Jr., and W.L. Root, Random Signals and Noise, McGraw-Hill (1958).
11. E.J. Hannan, Time Series Analysis, John Wiley (1960).



## DISTRIBUTION LIST

### Director's Office

W. H. Radford

### Division 2

F. C. Frick

### Division 3

V. J. Guethlen

L. A. Globus

E. Korngold (5)

P. R. Monson

B. Nanni

T. P. Tyler

### Division 4

R. P. Wishner

### Division 6

R. T. Prosser

### Group 21

J. Halberstein

F. Weinstein

### Group 22

V. A. Nedzel

W. I. Wells

W. Z. Lemnios

A. Freed

A. A. Grometstein

J. H. Hartogensis

L. J. Murray

J. H. Pannell

E. W. Pike

J. Rheinstein

J. Salerno

### Group 28

J. A. Arnow

J. F. Nolan

A. W. Armenti

C. R. Arnold(5)

F. Belvin

R. N. Davis

J. J. FitzGerald

L. A. Gardner, Jr.

H. K. Knudsen

O. A. Z. Leneman

J. B. Lewis

F. X. Mahoney

F. C. Schweppe

P. E. Wood, Jr.

### Group 41

E. M. Hofstetter

### Group 45

D. F. Delong

### Group 62

K. L. Jordan

### Group 63

R. Lerner

H. Sherman

### Group 64

P. E. Green, Jr.

P. L. Fleck

E. J. Kelly, Jr.

### Group 65

R. G. Enticknap

R. V. Wood, Jr.

### Group 66

T. J. Goblick

B. Reiffen

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)  Lincoln Laboratory, M.I.T.		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP None	
3. REPORT TITLE  A Program for the Estimation of Power Spectra			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Note			
5. AUTHOR(S) (Last name, first name, initial)  Arnold, Charles R.                      Korngold, Eric			
6. REPORT DATE 21 May 1965		7a. TOTAL NO. OF PAGES 56	7b. NO. OF REFS 11
8a. CONTRACT OR GRANT NO. AF 19 (628)-500		9a. ORIGINATOR'S REPORT NUMBER(S) TN 1965-16	
b. PROJECT NO.  c. None d.		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) ESD-TDR-65-75	
10. AVAILABILITY/LIMITATION NOTICES  None			
11. SUPPLEMENTARY NOTES  None		12. SPONSORING MILITARY ACTIVITY  Air Force Systems Command, USAF	
13. ABSTRACT  <p>A straightforward and general program of spectral analysis has been written in FORTRAN II for the IBM 7090 and is described in this report. The analysis is valid for sequences of equidistant data sampled from realizations of second order stationary stochastic processes. Alternatively, the program may be used to estimate the transfer function gain characteristic of a linear system on the basis of its sampled output.</p> <p>The Inputs to the program consists of the data sequence to be analyzed and FOUR control parameters. The Output consists of listings of the estimated values and of three CALCOMP plots of: the sample auto covariance (ACV) functions, the power spectral density (PSD), and a log-log plot of the PSD. The estimated PSD is consistent, being a periodogram smoothed with Hanning weights.</p>			
14. KEY WORDS  FORTRAN II                      power spectra IBM 7090			

Printed by  
United States Air Force  
L. G. Hanscom Field  
Bedford, Massachusetts

